BPS Wilson lines and backreacted geometries.

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Abstract

In this thesis we examine various aspects of supersymmetric Wilson lines. First we study Pestun's matrix model for the $\mathcal{N} = 2$ $N_f = 2N$ SCFT, and compute expectation values of Wilson loops. This informs us about possible holographic duals. We then turn to Wilson lines as a model for dense states in $\mathcal{N} = 4$ Super Yang-Mills theory. Working in the dual IIB string theory, we construct supersymmetric geometries which we expect to govern the stable $T = 0$ ground state of an $\mathcal{N} = 2$ QCD-like theory at finite baryon density. These exhibit a $z = 7$ Lifshitz scaling symmetry in the IR.

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted for any degree.

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Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Where correction services have been used, the extent and nature of the correction is clearly marked in a footnote(s). Other sources are acknowledged by footnotes giving explicit references. A bibliography is appended.

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To my parents

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Lastly and most importantly, to my dear Mum and Dad, to whom this thesis is dedicated, and about whom many theses could not say enough.

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Chapter 1

BPS Wilson lines

1.1 Supersymmetric gauge theories

Supersymmetric non-Abelian gauge theories in four dimensions have many nice properties. There is no space to give a complete overview here - for references see e.g. [6]. Here we simply discuss the material we will need in this thesis.

$\underline{\mathcal{N}=1}$

4D theories are built out of two types of multiplet: the vector multiplet, consisting (on-shell) of one gauge field and one Weyl spinor

$$\mathcal{V}: \qquad \{A_\mu, \quad \lambda\}$$

and the chiral multiplet, comprising one Weyl spinor and one complex scalar

$$\mathcal{X}: \qquad \{\lambda, \quad \Phi\}$$

These are representations of $\mathcal{N}=1$ SUSY (four real supercharges.) Gauge theories invariant under this supersymmetry can be formed by choosing a gauge group, and then minimally coupling some number of chiral multiplets to the vector multiplet in various representations. In addition, we can add gauge invariant interaction terms between the the chiral multiplets - these are encoded in the 'superpotential' $W(\mathcal{X}_i)$.

$\underline{\mathcal{N}=2}$

The $\mathcal{N}=2$ vector multiplet is one vector multiplet and one chiral multiplet in $\mathcal{N}=1$ language:

$$\mathcal{V}_{\mathcal{N}=2}: \qquad \{\mathcal{V}, \quad \mathcal{X}\}$$

The $\mathcal{N}=2$ 'hypermultiplet' is two $\mathcal{N}=1$ chiral multiplets of opposite chirality:

$$\mathcal{H}_{\mathcal{N}=2}: \qquad \{\mathcal{X}_{\text{left}}, \quad \mathcal{X}_{\text{right}}\}$$

$\underline{\mathcal{N}=4}$
On flat space, the $\mathcal{N}=4$ theory is completely specified by the gauge group, and goes by the name of $\mathcal{N}=4$ Super Yang-Mills (SYM) theory. Its field content is all in the same multiplet: one $\mathcal{N}=2$ vector multiplet and one hypermultiplet

$$\mathcal{N}=4 \text{ multiplet}: \qquad \{\mathcal{V}_{\mathcal{N}=2}, \quad \mathcal{H}_{\mathcal{N}=2}\}$$

Since this contains the gauge field, all its fields must be in the adjoint of the gauge group.

1.1.1 NSVZ beta function

$\mathcal{N}=1$ gauge theories in the absence of a tree-level superpotential, and all $\mathcal{N}=2$ gauge theories that we have described, are scale invariant at the classical level: their actions are invariant under a homogeneous rescaling of the 4D spacetime coordinates

$$x^\mu \mapsto \Omega^{-1}\, x^\mu$$

providing we also rescale the field variables, each by its appropriate 'scaling dimension' Δ_X

$$X \mapsto \Omega^{\Delta_X} X$$

where X represents any field. In particular, we can guarantee scale invariance by taking $\Delta_A = \Delta_{\phi_i} = 1$, $\Delta_{\lambda_i} = \frac{3}{2}$. However, at the quantum level the symmetry is broken. The functional integral has the effect of introducing scale dependence in the couplings, and renormalizing the fields as $X_{\text{bare}} = \sqrt{\mathcal{Z}_X}\, X_{\text{ren}}$. We must introduce a renormalization scale μ in a particular way ('scheme') - the details of the running will depend on the scheme. However the qualitative features, like the direction of running of coupling constants or the existence of fixed points, are scheme independent. We define the gauge coupling as the coefficient in the gauge action

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2g^2}\,\mathrm{Tr}\,\left((F_{\text{ren}})_{\mu\nu}(F_{\text{ren}})^{\mu\nu} + \cdots\right)$$

Given an $\mathcal{N}=1$ $SU(N)$ gauge theory coupled to various chiral multiplets $\mathcal{X}_i$ in $SU(N)$ representations R_i, remarkably [7, 8] it is possible to find a renormalization scheme in which the following β-function for the gauge coupling is exact to all orders in perturbation theory, and also non-perturbatively:

$$\beta_{8\pi^2/g^2} \equiv \frac{\partial}{\partial \log \mu}\left(\frac{8\pi^2}{g^2}\right) = \frac{3N - \sum_i T_{R_i}[1-\gamma_i]}{1 - g^2 N/8\pi^2} \tag{1.1}$$

which is the celebrated 'NSVZ beta function' (Novikov-Shifman-Vainshtein-Zakarov). In (1.1), T_{R_i} are the quadratic casimirs and γ_i are the anomalous dimensions of the chiral multiplets

$$\gamma_i \equiv -\frac{\partial \log \mathcal{Z}_{\Phi_i}}{\partial \log \mu} \quad .$$

We now describe in more detail the theories we will need.

1.1.2 $\mathcal{N}=4$ Super Yang-Mills theory

This theory was first considered in [9]. It is very convenient, especially for AdS/CFT purposes, to view this theory as coming from 10D SYM (one gauge field A_M and one Majorana-Weyl spinor Ψ) by dimensional reduction on T^6. The six real scalars ϕ_i are the internal components of A_M, and the single 10D spinor splits into four 4D Weyl spinors λ^I. Dimensional reduction breaks $SO(9,1) \to SO(3,1) \times SO(6)_R$, and in 4D $SO(6)_R \simeq SU(4)_R$ is the R-symmetry group under which the ϕ_i tranform in the **6** and the λ^I in the **4**. None of the fields have anomalous dimensions, so the NSVZ β-function with three adjoint chiral multiplets tells us the theory is conformal, which combines with the Poincaré SUSY to form a superconformal algebra. These transformations are all bundled into one 10D Majorana-Weyl spinor with linear coordinate dependence:

$$\epsilon = \epsilon_s + x^\mu \gamma_\mu \epsilon_c \tag{1.2}$$

where the constant MW spinors $\epsilon_{s,c}$ are the Poincaré and superconformal SUSY parameters, respectively. We use a chiral basis for the 10D Clifford algebra

$$\Gamma^M = \begin{pmatrix} & \tilde{\gamma}^M \\ \gamma^M & \end{pmatrix}$$

The on-shell SUSY transformations of the bosonic fields are

$$\delta A_\mu = \Psi \gamma_\mu \epsilon$$
$$\delta \phi_i = \Psi \gamma_i \epsilon$$

1.1.3 $\mathcal{N} = 2$ theories

We generate $\mathcal{N} = 2$ superconformal transformations, following [10], by restricting to $\gamma^{6789}\epsilon \equiv \gamma^6\gamma^7\gamma^8\gamma^9\epsilon = \epsilon$ in (1.2). Setting also $\epsilon_c = 0$ gives us the $\mathcal{N} = 2$ superpoincaré algebra. Setting $\gamma^{6789}\Psi = \Psi$ and throwing away the corresponding scalars gives pure $\mathcal{N} = 2$ SYM. Hypermultiplets in the fundamental of $SU(N)$ with masses can then be added in an $\mathcal{N} = 2$ covariant way, however this is all we need to know since Wilson lines will fall in the gauge sector, therefore to prove their supersymmetry we will not need the details. However we will sometimes be working in the Veneziano limit $N_f \sim N$, so the fundamentals will run in loops, contributing to the running of the gauge coupling and adding functional determinants to our formulae. $\mathcal{N} = 2$ theories have vanishing anomalous dimensions, so that if we add $N_f = 2N$ fundamental hypermultiplets the NSVZ formula is enough to show that the gauge coupling is exactly marginal - this theory is therefore believed to be conformal. It is sometimes known as the A_1 CFT, since it can arise as the low energy theory of a stack of D4 branes stretched between two NS5 branes - upon T-dualizing this setup we obtain an A_1-type singularity. This can be generalized to obtain a complete ADE-type class of theories.

1.2 Wilson lines

Wilson lines are natural operators to consider in any gauge theory. They are defined as the trace of the holonomy of the gauge field along a contour $\mathcal{C}$:

$$W_{\mathcal{R}}[\mathcal{C}] \equiv \mathrm{Tr}_{\mathcal{R}}\mathcal{H}(A, \mathcal{C}) = \mathrm{Tr}_{\mathcal{R}}\mathcal{P}\exp i\int_{\mathcal{C}} A$$

and are specified by $\mathcal{C}$ and a representation $\mathcal{R}$ of the gauge group[1]. When the contour is closed we refer to them as Wilson loops. Under a gauge transformation $U(x)$ the holonomy transforms by a 'conjugation' between the group element at either end of the contour:

$$\mathcal{H}(\mathcal{C}) \mapsto U(x_{\text{start}})\,\mathcal{H}(\mathcal{C})\,U^{-1}(x_{\text{end}})$$

therefore for a closed loop the trace makes W gauge invariant (at least under single-valued gauge transformations, see below). This is a useful property. We can in fact

[1]In this thesis we only consider $SU(N)$ gauge group.

reconstruct all local gauge invariant operators out of Wilson lines instead of the gauge field. This is what is done in the lattice formulation [11], where the fundamental variables are Wilson loops around 'plaquettes' (small squares with vertices at the lattice sites.) We here outline two salient uses of Wilson lines as probes of gauge theory

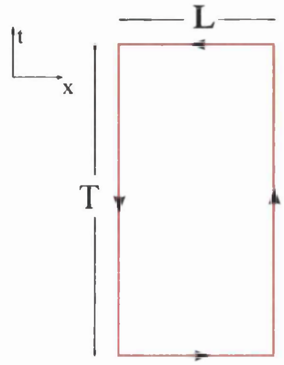


Figure 1.1: Rectangular Wilson line for the calculation of the $q\bar{q}$ potential.

dynamics.

The first is in the computation of the potential energy between an infinitely massive ('probe') static quark-antiquark pair. Consider the loop in figure 1.1 - the potential $V(L)$ is extracted as

$$\lim_{T\to\infty} W_{\Box}(\mathcal{C}) = \mathrm{e}^{-TV(L)}$$

The second example is this. In a thermal $SU(N)$ gauge theory with only adjoint fields (and therefore no 'quarks' as such), we can still define a notion of 'confined' and 'deconfined' phases. In such theories, we must integrate over 'large' gauge transformations

$$U_{\text{large}}(\tau) = \begin{pmatrix} \omega_k(\tau) & 0 & & & \\ 0 & \omega_k(\tau) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \omega_k(\tau)^{-(N-1)} \end{pmatrix}$$

where $\omega_k(\tau) \equiv \mathrm{e}^{2\pi i \frac{\tau}{\beta}\frac{k}{N}}$ with $k \in \mathbb{Z}$ and $\{S^1_\beta : 0 \le \tau < \beta\}$ the Euclidean thermal circle of the equilibrium QFT at temperature $1/\beta$. They have the property that $U(\tau+\beta) = g \cdot U(\tau)$, with $g \equiv \mathrm{e}^{2\pi i k/N}$ an element of the centre $\mathbb{Z}_N$ of $SU(N)$, and the adjoint fields are therefore single-valued. All local gauge-invariant operators are invariant under U_{large}, but the Wilson loop[1] around S^1_β transforms as

$$W_{S^1_\beta} \mapsto g \cdot W_{S^1_\beta}$$

(this is allowed to happen since U_{large} is not single-valued.) Its expectation value therefore serves as an order parameter for a transition between a $\mathbb{Z}_N$-symmetric and a $\mathbb{Z}_N$-broken phase.

The connection with confinement is that $W_{S^1_\beta}$ is the free energy $\mathrm{e}^{-\beta F}$ of an isolated probe fundamental quark, inserted by hand into the theory. A $W_{S^1_\beta} = 0$ phase corresponds to infinite free energy, so that isolated quarks are always thermodynamically disfavoured and there is confinement. In a $W_{S^1_\beta} \neq 0$ phase where the centre is broken, isolated quarks have finite free energy and the theory 'deconfines'.

1.3 Wilson lines in supersymmetric gauge theories

In most[2] supersymmetric gauge theories, $W_{\mathcal{R}}[\mathcal{C}]$ is an operator which does not preserve any SUSYs for generic $\mathcal{C}$. However, if the SUSY is sufficiently extended, the vector multiplet will contain adjoint scalars. If we view the theory as a dimensional reduction from a higher dimensional theory, these can be viewed as the components of the gauge field along the reduction coordinates. Therefore their SUSY transformations 'match' those of the gauge fields in a sense to be made explicit later, and by adding specific scalar couplings in the exponent of the loop we can preserve a fraction of the supersymmetries. Consider the generalized Wilson line

$$W_{\mathcal{R}}[\mathcal{C}] \equiv \mathrm{Tr}_{\mathcal{R}} \mathcal{H}(A, \mathcal{C}) = \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \int_{\mathcal{C}} (i\, \dot{x}^\mu A_\mu + n^i \phi_i) \mathrm{d}s \tag{1.3}$$

1.3.1 $\mathcal{N} = 4$ theory

Here $i = 1, \cdots 6$. Taking the variation of the integrand in (1.3) gives (setting $\epsilon_c = 0$ for now)

$$\dot{x}^\mu \gamma_\mu \epsilon_s + n^i \gamma_i \epsilon_s \stackrel{!}{=} 0$$

[1]The Wilson loop around the thermal circle is often called the 'Polyakov loop'.

[2]Sometimes the Wilson loop with no scalar coupling can be considered to be supersymmetric, for example in pure Chern-Simons theory.

for a BPS loop. This projection guarantees SUSY locally at a point on the contour. However we will in general need to impose an infinite number of projections, one for each point on the loop. For the whole loop to be BPS, we must have at most 4 independent projectors, which all commute. In [12] this is solved by setting

$$n^i = M^i_\mu \dot{x}^\mu \tag{1.4}$$

for a *constant* 6×4 matrix M^i_μ. Then for the whole loop we need to impose only 4 constant projections

$$(\gamma_\mu + M^i_\mu \gamma_i)\epsilon_s = 0 \qquad \mu = 0, 1, 2, 3$$

These imply that $M^i_\mu M^i_\nu = \eta_{\mu\nu}$, i.e. when viewed as vectors in $\mathbb{R}^6$ the M_μ are orthonormal. For a generic contour we must also restrict to $\epsilon_c = 0$.

For general contours we must impose 4 projections, preserving one real supercharge ('$\frac{1}{16}$-BPS'). But if the contour is restricted to a linear subspace we need fewer projections. For example if $\dot{x}^3(s) = 0 \ \forall s$ then we do not need the $\mu = 3$ condition, and preserve 2 charges. A planar loop inside some $\mathbb{R}^2 \subset \mathbb{R}^4$ preserves 4, and a straight line has constant scalar coupling[1] and preserves 8 (we will see later that for this loop the SUSY is enhanced again.) Prettily, it was shown in [12] that one may expect all the $\frac{1}{2}$- and $\frac{1}{4}$-BPS loops within this ansatz to have $\langle W \rangle = 1$ exactly.

1.3.2 Circular loops and loops on S^3

This does not exhaust the list of SUSY Wilson loops. In particular one can consider loops invariant under some superconformal generators $\epsilon_c \neq 0$. Into this class falls the circular spatial loop with constant scalar coupling:

$$\mathcal{C}: \qquad \begin{array}{ll} x^1 = r_0 \cos\theta & x^3 = 0 \\ x^2 = r_0 \sin\theta & x^0 = 0 \end{array} \qquad n = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The variation of this gives

$$\epsilon_s = r_0 \, i \gamma^4 \gamma^1 \gamma^2 \epsilon_c \tag{1.5}$$

[1] This is known as the 'Maldacena-Wilson line'.

This last case is one of a further class of loops introduced in [13]. Their contours are all contained within a spatial S^3 subspace. We decompose the motion along $\mathcal{C}$ into the right-invariant one-forms, and use these components as the couplings for three of the scalars, say $\phi_{1,2,3}$. That is we take

$$\mathcal{W} = \frac{1}{\dim[\mathcal{R}]} \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \int_{\mathcal{C} \subset S^3} \left(i\, \mathrm{d}x^\mu A_\mu + \frac{1}{2} \sigma_a^R N^{a\,i} \phi_i \right)$$

where σ_i^R are the right-invariant one-forms, and without loss of generality we take

$$(N^{a\,i}) = \begin{pmatrix} 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{pmatrix}$$

These loops will generically have $\langle W \rangle \neq 1$. Note that this applies therefore to the circular loop, even though it is related to the straight line by a conformal transformation. This is because the mapping is ill-defined at the point $|x| = \infty$.

1.3.3 Parallel straight lines

As a final example which will be relevant later, let us consider the SUSYs preserved by two parallel straight lines lying along the t axis with $x^{2,3} = 0$, one at $x^1 = 0$, the other at $x^1 = x$. For $x^1 = 0$ we find

$$(\gamma^0 + \gamma^4)\epsilon_s + t(1 - \gamma^4\gamma^0)\epsilon_c \stackrel{!}{=} 0 \quad \forall t$$

satisfied by ϵ_s and ϵ_c with the projections $(\gamma^4\gamma^0 + 1)\epsilon_s = 0$, $(\gamma^4\gamma^0 - 1)\epsilon_c = 0$, giving 16 supercharges.

For the $x^1 = x$ line the projection conditions have an extra piece:

$$(\gamma^0 + \gamma^4)\epsilon_s + t(1 - \gamma^4\gamma^0)\epsilon_c - x\,\gamma^1(\gamma^0 + \gamma^4)\epsilon_c \stackrel{!}{=} 0 \quad \forall t, \quad \text{fixed}\, x$$

This is solved by the same projections as the other line, except now the preserved spinors are ϵ_s (with $\epsilon_c = 0$) and the linear combination

$$\epsilon_c = -\frac{1}{2x}(1 + \gamma^4\gamma^0\gamma^1)\epsilon_s \quad ,$$

which is a rotated version of the superconformal paramter for the $x^1 = 0$ line. It is different for every x, so that only the 8 real Poincaré SUSYs $(\gamma^4\gamma^0 + 1)\epsilon_s = 0$ are preserved by the pair of lines, and the same applies to an array of any number of parallel lines.

Note also that for a single line, the SUSY projector is left invariant if we flip both the orientation of the line and the sign of the scalar coupling. Thus two *anti-parallel* lines at the same point preserve the same SUSYs as a single line.

1.4 SUSY lines in $\mathcal{N} = 2$ theories

As mentioned in section 1.1, the $\mathcal{N} = 2$ vector multiplet is a restriction of the $\mathcal{N} = 4$ version, so as long as we stick to the vector multiplet many of the results carry over to any $\mathcal{N} = 2$ theory. In particular the ansatz involving $N^{a\,i}$ still works, but now since we only have two real adjoint scalars we must restrict to two of the right-invariant one-forms σ^R_a. In any case, the circular loop (a great circle of the S^3) imposes the same number of projections and so preserves 8 of the 16 superconformal SUSYs. This loop will be the subject of chapter 2.

1.5 Holographic duals of Wilson lines

In this thesis we will be concerned largely with the role of Wilson lines in the AdS/CFT correspondence, which is by now well known and passes many highly non-trivial tests at both strong and weak 't Hooft coupling $\lambda \equiv g^2_{YM}N$ [14–16]. The simplest case is the mapping between $SU(N)$ $\mathcal{N} = 4$ SYM on flat space and type IIB string theory with $AdS_5 \times S^5$ boundary conditions. This conjecture arises by considering the physics very close to a stack of N parallel D3 branes, which leads one to identify the low energy field theory living on the branes with gravity in the near-horizon geometry.

Since a fundamental string ending on one of the D3s is decribed on the worldvolume as the insertion of an operator $\mathrm{Tr}_\square \exp(i\int_\mathcal{C} A + \cdots)$, it is natural to identify F1s with certain boundary conditions at the AdS boundary as Wilson lines in $\mathcal{N} = 4$ in the fundamental representation. It is generally expected that this carries over to other examples of gauge/string duality. At large λ, when the coupling of the string worldsheet theory is small, the leading terms in the path integral come from saddle point configurations with Dirichlet boundary conditions tracing out the contour $\mathcal{C}$ on the boundary. The unit scalar coupling in $\mathbb{R}^6$ which appears in SUSY loops is identified as the path traced on S^5. We may also impose $SO(6)$-invariant Neumann boundary conditions, and there is evidence [17, 18] that this loop is dual to the pure Wilson line without scalar coupling. In [18], a holographic RG flow is constructed between the pure gauge loop in the UV and the SUSY loop in the IR.

This correspondence works for loops in the fundamental representation. What are the duals of loops in other representations? We now describe what is known about this. We deal here with the $\frac{1}{2}$-BPS straight Wilson line studied in the last section[1]. Different representations of $SU(N)$ can be specified by a *Young tableau* [19]. Loops have different types of description depending on the order of magnitude of the number of boxes in the Young tableau for $\mathcal{R}$, or equivalently its dimension $\dim[\mathcal{R}]$.

1.5.1 Probe branes

If $\dim[\mathcal{R}] \sim \mathcal{O}(1)$, the loop is described by several coincident F1s, appropriately symmetrized. For $\dim[\mathcal{R}] \sim \mathcal{O}(N)$, the dual description is in terms of D3 and D5 branes with worldvolume flux. These can be understood as follows: in a process analogous to the brane dielectric effect [20], the dynamics of many coincident strings may induce some D-brane charge. As such they must 'blow up' into a higher-dimensional brane, which must still carry the original string charge - this is now realized as a non-vanishing worldvolume gauge field F. These branes wrap contractible cycles in $AdS_5 \times S^5$, but are stablilized by F [21].

In particular, a loop in the rank-k symmetric representation S_k is dual to a D3 on $AdS_2 \times S^2 \subset AdS_5$. We choose the '$z$' type AdS_5 coordinates, in which the boundary corresponds to $z = 0$ (we set the AdS radius $L = 1$):

$$\mathrm{d}s^2_{AdS_5} = \frac{-\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\psi^2 + \sin^2\psi\,\mathrm{d}\phi^2)}{z^2} + \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_4^2\right) \qquad (1.6)$$

and parametrize the S^4 with angles ϕ^i, $i = 1, 2, 3, 4$. Then explicit embedding is desribed in the box. It looks like an S^2 which 'fans out' into the boundary directions as it comes away from $z = 0$, although the gravitational warping means that covariant size of this S^2 is actually constant. Importantly, this worldvolume is entirely on the non-compact AdS part of the geometry.

[1]The corresponding story for more general lines has not been widely studied. This would be an interesting avenue for research, although challenging because of the reduced symmetry of the problem.

The antisymmetric representation (A_k) loop is a D5 on $AdS_2 \subset AdS_5$ and $S^4 \subset S^5$. In constrast to the symmetric case, this brane has its worldvolume partly on the compact S^5 part of the geometry. The S^4 is embedded along a latitude of S^5, as described by a constant-θ slice in (1.6). The latitude θ_k of the slice is determined by the rank of the representation, as follows. Totally antisymmetric representations of $SU(N)$ cannot be arbitrarily large, in the sense that the rank-N representation is equivalent to the singlet, as can be seen by contracting with the invariant completely antisymmetric tensor $\epsilon^{a_1 \cdots a_M}$. This means there cannot be more than $N-1$ rows in the Young tableau (this contrasts with the symmetric case, since a single row can be arbitrarily long (see figure 1.2.) This means that as we take $k, N \to \infty$ with the ratio $f \equiv k/N$ (which is held fixed in the limit) takes continuous values in the range $[0,1)$, which is mapped to the finite range of the θ coordinate. In particular, loops with small f are mapped to branes near the North pole of S^5. As f increases, the brane is pulled down over the sphere, and at f just below one the brane is near the South pole. The equation mapping rank to latitude is $\theta_k - \sin\theta_k \cos\theta_k = \pi f$.

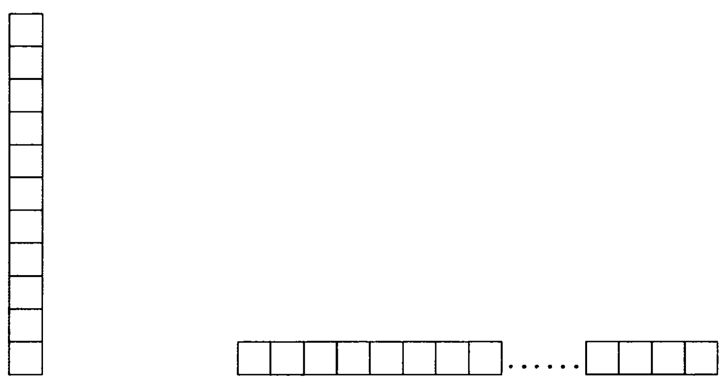


Figure 1.2: The antisymmetric (left) vs. the symmetric (right) representations of $SU(N)$: whereas a column can be at most $N-1$ boxes tall (if we want to avoid redundancy), a row can go on forever.

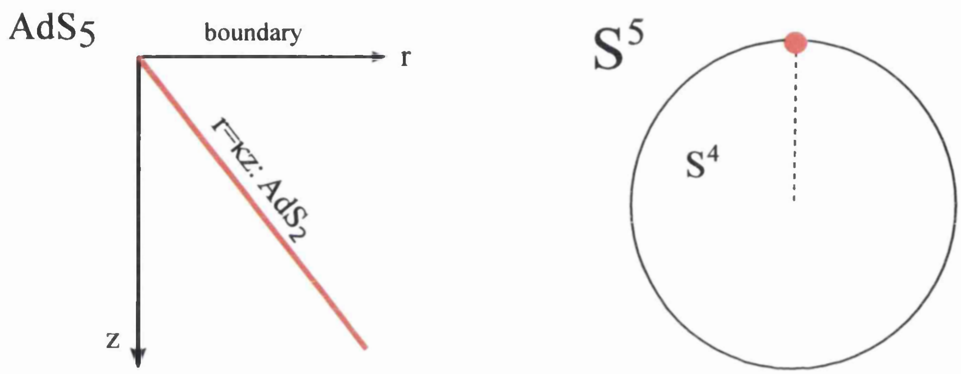
In each case, symmetric and antisymmetric, the $SL(2,\mathbb{R})$ subgroup of the conformal transformations preserved by the straight Wilson line is realized through the $SL(2,\mathbb{R})$ isometry of the AdS_2 factor in the worldvolume, and a non-zero flux $F_{0r} \equiv F \neq 0$ indicates the dissolved F1 sources. These embeddings are described in their respective boxes., where we also note the preserved supercharges of the configurations, which agree with the field theory. The matching between the supercharges on either side of the duality is worked out in appendix B - the spinors $\epsilon_{s,c}$ make reference to this appendix; $\mathcal{K}$ and $\mathcal{I}$ are complex conjugation and multiplication by i, respectively.

D3

In the coordinates (1.6), the brane embedding is (in units $2\pi\alpha' \equiv 1$, $\kappa \equiv \frac{\sqrt{\lambda}}{4}\frac{k}{N}$):

$$z = \sigma^1 \quad t = \sigma^0 \quad r = \kappa z$$
$$\psi = \sigma^3 \quad \phi = \sigma^4$$

where $\sigma^{0,1,2,3}$ are the worldvolume coordinates.



Calculating the induced metric, we find

$$\mathrm{d}s^2_{\mathrm{ind}} = (1+\kappa^2)\left(\frac{-(\mathrm{d}\sigma^0)^2 + (\mathrm{d}\sigma^1)^2}{(\sigma^1)^2}\right) + \kappa^2 \mathrm{d}\Omega_2^2$$

which is the product of AdS_2 (radius $\sqrt{1+\kappa^2}$) and an S^2 (radius κ).

SUSY conditions: We use the kappa projection conditions for the D-brane to see which supersymmetries of the background are preserved by this embedding:

$$\Gamma_\kappa = \frac{1}{\sqrt{1-F^2}}\left[(z'(r)\Gamma_{\hat{0}\hat{z}\hat{\psi}\hat{\phi}} + \Gamma_{\hat{0}\hat{r}\hat{\psi}\hat{\phi}})\mathcal{I} - F\Gamma_{\hat{\psi}\hat{\phi}}\mathcal{K}\mathcal{I}\right]$$

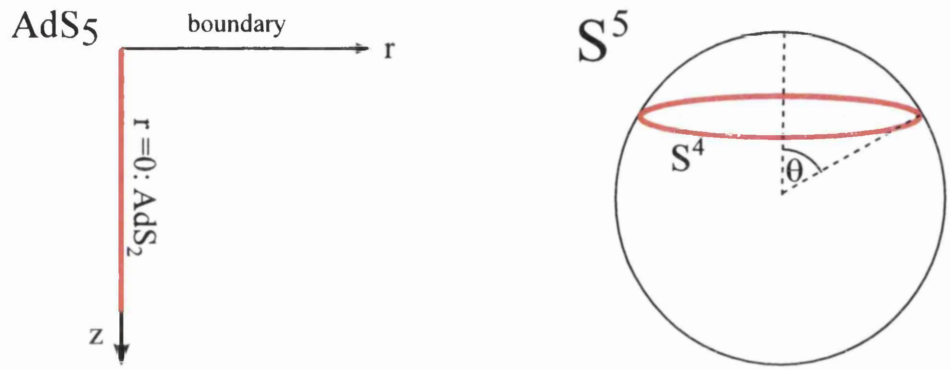
$\Rightarrow \gamma^{\hat{r}\hat{0}}\epsilon_s = 0\,, \quad \gamma^{\hat{r}\hat{0}}\epsilon_c = +\epsilon_c.$

D5

Embedding:

$$z = \sigma^1 \quad t = \sigma^0 \quad x^i = 0$$
$$\theta = \theta_k = \text{constant} \quad \phi_a = \sigma^a$$

where θ_k is related to the rank of A_k by $\quad \theta_k - \sin\theta_k \cos\theta_k = \pi \frac{k}{N}$.



SUSY conditions:

$$\Gamma_\kappa = \frac{1}{\sqrt{1-F^2}} \left[\Gamma_{\hat{t}\hat{z}\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4} \mathcal{K}\mathcal{I} + F\,\Gamma_{\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4} \mathcal{I} \right]$$

$\Rightarrow \gamma^{\hat{r}\hat{0}} \epsilon_s = 0\,, \quad \gamma^{\hat{r}\hat{0}} \epsilon_c = +\epsilon_c.$

1.5.2 Bubbling geometries

For generic representations in which $\dim[\mathcal{R}] \sim \mathcal{O}(N^2)$ or higher, the actions of the probe D3/D5 branes become $S_{DBI+WZ} \sim \mathcal{O}(N^2)$ and backreact on the $AdS_5 \times S^5$ geometry. The description must now be a smooth solution of IIB supergravity with the appropriate symmetries and fluxes. These were constructed in an impressive series of papers [4, 22, 23]. The idea in constructing these geometries is represent the symmetries of the Wilson loop as isometries of the spacetime: $SL(2,\mathbb{R})$ becomes AdS_2, the spatial rotational symmetry $SO(3)$ becomes S^2, and the $SO(5)$ preserved R-symmetry becomes an S^4. In order to incorporate these we make an ansatz for the geometry as a fibration of all of these spaces over a 'base' Riemann surface Σ: The metric ansatz is

$$\mathrm{d}s_{10}^2 = \mathrm{e}^{2A} \mathrm{d}s_{AdS_2}^2 + \mathrm{e}^{2B} \mathrm{d}s_{S^2}^2 + \mathrm{e}^{2C} \mathrm{d}s_{S^4}^2 + h^2(\mathrm{d}y^2 + g^2 \mathrm{d}x^2) \tag{1.7}$$

where A, B, C, g and h are functions of x and y, i.e. we have the three factor spaces fibered over $\Sigma(x, y)$. We also make appropriate ansatze for the field strengths

$$F_5 = (1 + *)\mathrm{d}f \wedge vol(S^4) \tag{1.8}$$

$$F_3 = f_3 \wedge vol(AdS_2) \tag{1.9}$$

$$H_3 = h_3 \wedge vol(S^2) \tag{1.10}$$

$$\phi = \phi(x, y) \qquad C^{(0)} = 0 \tag{1.11}$$

It so happens that by plugging these ansatze into the supersymmetry variations ('BPS equations') of IIB supergravity and solving them, we automatically solve all the equations of motion and Bianchi identities. The BPS equations reduce to a set of non-linear differential equations on Σ. Remarkably, it is possible to linearize them [23] and therefore to find a complete solution to the problem of backreacting Wilson lines.

The emergent picture is this - the whole solution is determined by two real harmonic functions h_α, h_β on Σ. An important role is played by the boundary of Σ. Since the imaginary and real parts of any holomorphic function are harmonic, we may always choose our coordinate $z \equiv x + iy$ on Σ such that $y = h_\beta$, and we then choose the real axis $h_\beta = 0$ to be the boundary of Σ, such that Σ is the upper half-plane. This boundary is not to be confused with the boundary of AdS_5 - rather it is a certain privileged line inside the geometry which will in general only intersect the AdS boundary at one point, which we can take to be the point at infinity. We have now fixed some of the diffeomorphism invariance, and the solutions are described completely by h_α. In order to have a smooth geometry, we must impose either vanishing Dirichlet (D) or Neumann (N) boundary conditions on h_α on the $y = 0$ line.

The product of the radii of the two spheres vanishes on the real axis: $\mathrm{e}^{B+C} \propto h_\beta$, therefore along this line either the S^2 or the S^4 must shrink to zero size. Also, h_α is proportional to e^C times some never-zero function. Therefore for (D) segments of the axis (when h_α vanishes) the S^2 radius e^B is non-vanishing, while for (N) segments S^4 is non-vanishing. If we take paths straddling these regions, as in figure 1.3, we find non-contractible S^3 and S^5 homology cycles[1]. These homology cycles are like 'holes' in the geometry, so the backgrounds have been dubbed 'bubbling'. The integrals of F_3 and F_5

[1]This is because fibering an S^{d-1} over a line interval such that its radius only vanishes at the end points produces a space that is topologically an S^d.

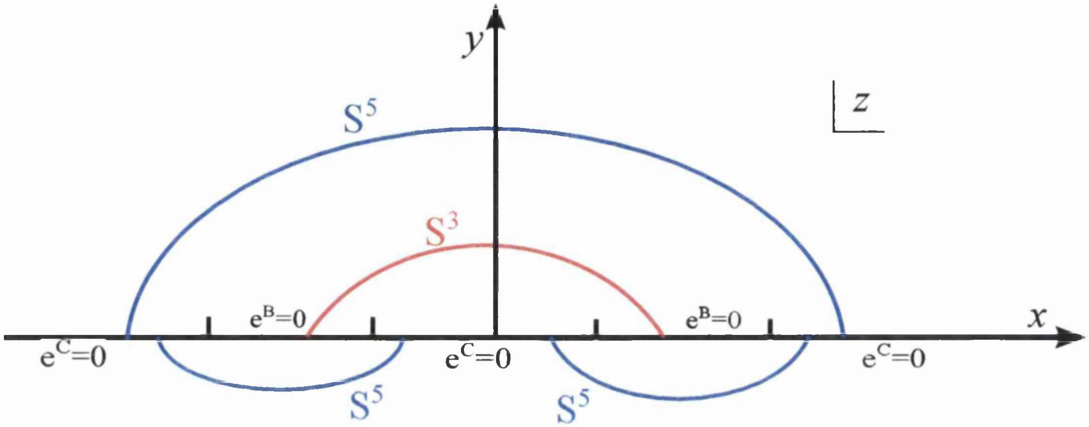


Figure 1.3: The homology cycles of the 'bubbling' geometries corresponding to straight Wilson lines in $\mathcal{O}(N^2)$ representations. Here we picture a representation involving a rectangular tableau as in figure 1.4.

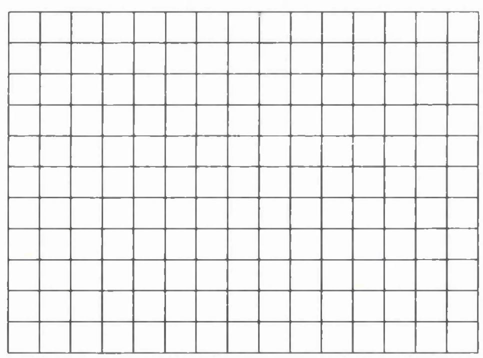


Figure 1.4: Rectangular Young tableau corresponding to the bubbling geometry in figure 1.3. The lengths of the sides should be scaled up to $\mathcal{O}(N)$.

over these cycles are related to the sizes of different blocks in the corresponding Young tableau - for example, in figures 1.3 and 1.4, the width of the tableau is proportional to the integral of H_3 over the S^3, while its height is proportional to the integral of F_5 over the right of the two smaller S^5's.

1.5.3 Circular loop expectation values

We note that all of the above straight line D branes have vanishing on-shell action, confirming field theory expectations. The on-shell action for these so-called 'bubbling geometries' does not seem to have been calculated, but the expected value is $S_{\text{on-shell}} =$

0. Some progress in this direction was made in [24]. There are two subtleties: finding 10-dimensional counterterms, and knowing which action to use for the self-dual five-form F_5.

As noted before, upon conformal mapping to a circle the expectation values of these loops pick up an anomalous contribution from infinity. The corresponding string/D brane embeddings are known [25] (they are obtained by performing an $SO(4,2)$ transformation in AdS_5 on the straight line embeddings), and they have non-zero actions. As an example, let us take the fundamental representation, dual to a fundamental string (F1). For a circular loop of radius R, in coordinates (1.6) we find the embedding

$$t=0\,,\quad z=\sqrt{R^2-r^2}\,,\quad r=\sigma^1\,,\quad \psi=\frac{\pi}{2}\,,\quad \phi=\sigma^0\,,\quad \theta=\phi^i=0$$

which solves the equations of motion for the Nambu-Goto action. The on-shell action is

$$\begin{aligned} S_{NG} &= T_{F1}\int_\Sigma \sqrt{\det g_{\mathrm{ind}}} = T_{F1}\int \mathrm{d}r\mathrm{d}\phi\,\frac{r}{z(r)^2}\sqrt{1+z'(r)^2} \\ &= -2\pi R\,T\int_\epsilon^R \frac{\mathrm{d}z}{z^2} = (-2\pi+\mathrm{divergent})\,T\,L^2 \end{aligned}$$

where we have restored the AdS radius L in the final expression. Thus, using the saddle point technique and adding a counter-term to cancel the divergent piece in action, we find a prediction for the expectation value of the circular fundamental loop at large λ:

$$\langle W_\square\rangle = \exp(-S_{NG}^{\mathrm{on\ shell}}) = \exp(2\pi\,L^2\,T_{F1}) = \exp(\frac{L^2}{l_s^2}) = \mathrm{e}^{\sqrt{\lambda}} \tag{1.12}$$

Where we used the string length $l_s\equiv\sqrt{\alpha'}$.

There are corresponding calculations for circular versions of the D3/5 branes described above - we only quote them here. For the antisymmetric case A_k, in the $\lambda\gg 1$ limit, this takes the value

$$W\sim \mathrm{e}^{\frac{2}{3\pi}N\sqrt{\lambda}\sin^3\theta_k} \tag{1.13}$$

while for S_k we have

$$W\sim \mathrm{e}^{N\frac{\lambda}{8}(\frac{k}{N})^2}\quad .$$

As in the straight line case, the D5 brane embedding for the circular case is specified by the rank k/N alone, therefore the dependence on λ and N in (1.13) can be worked out by simple considerations: it is the exponential of the action of a D brane $S \sim 1/(g_s l_s^6)$, and all of the brane dimensions are fixed by the AdS radius L. Therefore the expectation value must take the form $\exp(\frac{(L/l_s)^6}{g_s} g(f)) = \exp(N\sqrt{\lambda}\, g(f))$, for some function $g(f)$. In particular note that in this expression the size of the brane in string units is measured by the parametric dependence of the exponent on λ.

1.6 Localization and matrix models

We now describe the remarkable work done by Pestun in [10]. It turns out that when an $\mathcal{N} = 2$ gauge theory is placed on a (round) Euclidean S^4, Wilson loops preserving a certain supercharge can be calculated exactly using a $(0 + 0)$-dimensional matrix model. That this is so can be deduced using the old technique of localization of the path integral ([26] and references therein.)

1.6.1 General localization argument

The basic idea is the following. Let there be a QFT invariant under a fermionic supercharge Q which squares to some $U(1)$ bosonic symmetry. The action S is invariant, as is the measure $\mathcal{D}\phi$, (ϕ respresents all fields.) Consider the expectation value of an operator $\mathcal{O}$ which is Q-invariant:

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi \, \mathcal{O} \, \mathrm{e}^{-S}$$

Now let us deform the action by a positive-definite Q-exact term, with some coefficient $t \in \mathbb{R}^+$

$$\langle \mathcal{O} \rangle_t = \int \mathcal{D}\phi \, \mathcal{O} \, \mathrm{e}^{-S-t\{Q,V\}}$$

then differentiate with respect to t:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle \mathcal{O} \rangle_t &= -\int \mathcal{D}\phi \, \mathcal{O} \, \{Q, V\} \, \mathrm{e}^{-S-t\{Q,V\}} \\
&= -\int \mathcal{D}\phi \left\{ Q, \mathcal{O} V \mathrm{e}^{-S-t\{Q,V\}} \right\} = 0
\end{aligned}$$

The last equality follows since the integral is a total variation in field space, and holds if appropriate boundary conditions are specified. Thus $\langle \mathcal{O} \rangle_t = \langle \mathcal{O} \rangle \ \forall t$. The idea is now to send $t \to \infty$, in which limit we may use a saddle point approximation to the path integral. Since $\{Q,V\} \geq 0$, the saddle points are those field configurations giving $\{Q,V\} = 0$. We say the integral has 'localized' onto this locus. The standard prescription tells us to evaluate the exponent at the saddles, multiply by the fluctuation determinant of the fields, and then sum over saddles.

1.6.2 The case of SUSY Yang-Mills on S^4

Pestun begins by considering $\mathcal{N} = 4$ SYM on a Euclidean S^4 of radius R. The compact Euclidean space is convenient for the purposes of calculating functional determinants, since then the spectra of kinetic operators will typically be positive definite and discrete[1]. He formulates the theory as a dimensional reduction from 10D $\mathcal{N} = 1$ SYM on T^6 as in section 1.1, regarding $x^{1,2,3,4}$ as the four unreduced coordinates. We have the bijection $\mathbb{R}^4 \leftrightarrow S^4$, by viewing $\mathbb{R}^4$ as embedded in $\mathbb{R}^5(X^{1,2,3,4,5})$ with $(X^5 - R)^2 + \sum_{I=1}^4 (X^I)^2 = R^2$ and considering the stereographic projection (as depicted in figure 1.5):

$$x^\mu = \frac{X^\mu}{1 - \frac{X^5}{2R}} \qquad .$$

In this mapping, $|x| = 2R$ maps to the equator. Note that locally it is a conformal transformation, since $g_{S^4} = \frac{1}{(1+\frac{x^2}{4R^2})^2} g_{\mathbb{R}^4}$. However there are global issues, since this conformal factor vanishes as $x^\mu \to \infty$ so that all points on a large $S^3 \subset \mathbb{R}^4$ map to a single point (the North pole.)

The SUSY parameter ϵ must be a conformal Killing vector on S^4, which we can write as

$$\epsilon = \frac{1}{\sqrt{1+\frac{x^2}{4R^2}}}(\epsilon_s + x^\mu \gamma_\mu \epsilon_c)$$

This satisfies

$$\nabla_\mu \epsilon = \tilde{\epsilon} \qquad , \qquad \tilde{\epsilon} \equiv \frac{1}{\sqrt{1+\frac{x^2}{4R^2}}}(\epsilon_c - \frac{1}{4R^2} x^\mu \gamma_\mu \epsilon_s) \tag{1.14}$$

[1] Note that although the Majorana-Weyl condition is not consistent Euclidean space, and we therefore must consider complexified spinors, we can instead think of integrating over two independent spinors Ψ, $\bar{\Psi}$, but taking the contour to be such that $\Psi = \bar{\Psi}$.

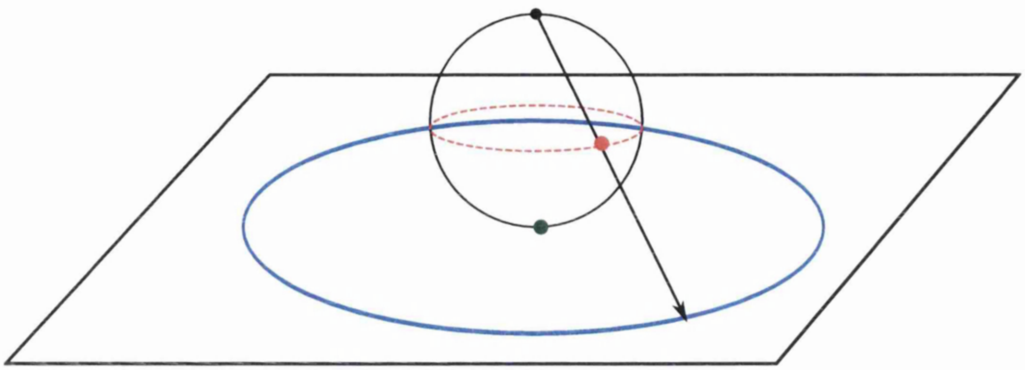


Figure 1.5: The stereographic projection: the S^4 sits with its South pole resting at the origin of $\mathbb{R}^4$ (green). A point on the equator (red dotted) of the sphere is mapped to a point on a circle (blue) of radius $2R$ on the plane.

Since the localization argument involves the SUSY invariance of fields *inside the path integral*, it is necessary to close the supersymmetry transformation Q *off-shell*. While it is not known how to close supersymmetry algebras off-shell for more than four supercharges using a finite number of auxiliary fields[2], there does exist a prescription, due to Berkovits [27], to close the 10D $\mathcal{N}=1$ (equivalently the 4D $\mathcal{N}=4$) algebra off-shell for any one given supercharge. Seven real auxiliary scalar fields K_A $(A = 1, \cdots 7)$ are necessary. The action takes the form

$$S = -\frac{1}{g_{YM}^2}\int \mathrm{d}^4x\, \mathrm{tr}\left(\frac{1}{2}F^{MN}F_{MN} - \Psi\gamma^M D_M\Psi + \frac{2}{R^2}\phi_i\phi^i + K^AK^A\right) \tag{1.15}$$

and is invariant off-shell under the supersymmetry transformations

$$\begin{aligned}
\delta_\epsilon A_M &= \Psi\gamma_M\epsilon \\
\delta_\epsilon\Psi &= \frac{1}{2}\gamma^{MN}F_{MN} + \frac{1}{2}\gamma^{\mu i}\phi_i \mathrm{D}_\mu\epsilon + K^A\nu_A \\
\delta_\epsilon K_A &= -\nu_A\gamma^\mu \mathrm{D}_\mu\Psi
\end{aligned}$$

where ν^A are seven MW spinors satisfying

$$\begin{aligned}
\epsilon\,\gamma^M\nu_A &= 0 \\
\nu_A\gamma^M\nu_B &= \delta_{AB}\,\epsilon\,\gamma^M\epsilon
\end{aligned}$$

[2]But using 'harmonic superspace', which contains an infinite number of fields, we can close the algebra for 8 supercharges simutaneously. The complete off-shell closure of the 16 Poincaré SUSYs of the $\mathcal{N}=4$ theory is, to the best of the author's knowledge, still an open problem.

Provided these conditions on the ν_A are satisfied, the supersymmetry algebra closes. It can be shown that for any given ϵ we can always find seven such spinors ν_A, but their form will not be relevant to our computations - suffice to say that they exist. Also note that, although the SUSY transformation look like they are indepedent of R, R does appear in $\delta_\epsilon \Psi$ because of (1.14) - indeed the variation of the conformal scalar mass term is cancelled by a term in the variation of the fermion kinetic term.

The supersymmetric operator $\mathcal{O}$ whose expectation value we will be interested in computing is the Wilson loop around the equator with constant unit scalar coupling. This is defined by setting $r_0 = 2R$ in the flat space circular loop of section 1.3.2 and mapping to the equator of S^4 using the stereographic projection. Therefore the supercharge Q which is used in the localization should satisfy the projections (1.5) for that loop - this is condition (a) below. Explicitly, we demand that Q being generated by a SUSY parameter ϵ subject to the following conditions:

(a) $\epsilon_c = \frac{1}{2R}\gamma^{012}\epsilon_s$

(b) $\gamma^{5678}\epsilon_s = \epsilon_s$

(c) $\gamma^{1234}\epsilon_s = -\epsilon_s$

In this notation we have reduced over $x^{5,6,7,8,9}$ and Euclideanized coordinate x^0, producing scalars $\phi_{0,5,6,7,8,9}$. So sending $\gamma^0 \mapsto -i\gamma^0$ and comparing with (1.5), (a) are the SUSYs preserved by a circular loop of radius $2R$, i.e. equatorial on S^4, (b) restricts to the $\mathcal{N} = 2$ superconformal algebra, and (c) fixes the chirality of ϵ so that the algebra closes onto the $SO(5)$ isometries of S^4 rather than the full $SO(5,1)$ conformal group - this means that localization is compatible with an $\mathcal{N} = 2$ mass deformation.

As mentioned in section 1.1.3, we can move from $\mathcal{N} = 4$ SYM to pure $\mathcal{N} = 2$ SYM by restricting the spinor to $\gamma^{6789}\Psi = \Psi$. We may then add hypermultiplets in any given dimension of $SU(N)$ - we do not write down the action or field transformations here. The only place in which these fields enter is in the 1-loop determinant, which we discuss later.

The preceding restricts ϵ to four real independent components. In order to completely specify it, we require that its bilinear $v^M \equiv \epsilon\gamma^M\epsilon$ generates in its spactime (μ) components equal $U(1)$ rotations in the (x^1–x^2) and (x^3–x^4) planes. The internal components should be

$$\begin{aligned} v^0 &= i \\ v^9 &= -\cos\theta \\ v^{5,6,7,8} &= 0 \end{aligned}$$

and the normalization is such that $\sqrt{v^\mu v_\mu} = \sin\theta$, where θ is the polar angle on S^4. The i comes from the Euclidean signature, and does not cause problems.

We must now chose a Q-exact localizing term - in Pestun's case it is $V = \mathrm{Tr}\left(\Psi\left\{Q,\Psi\right\}\right)$. Thus the bosonic part of the deformation is positive definite:

$$\left\{Q,V\right\}|_{\text{bosonic}} = \mathrm{Tr}\left(\left\{Q,\Psi\right\},\left\{Q,\Psi\right\}\right)|_{\text{bosonic}} \geq 0$$

The locus of saddles then consists of those configurations for which $\{Q,\Psi\} = 0$. This condition turns out to be very restrictive. The result is that, if we only consider smooth field configurations, we must have

$$\begin{aligned} \phi_0 &= \text{constant} \equiv a \\ K_A &= 2(\nu_A\tilde{\epsilon})a \\ \text{all other fields} &= 0 \end{aligned}$$

The path integral therefore reduces to a integral over the entries of the adjoint scalar zero mode a, i.e. a zero-dimensional matrix model. Since the original path integral had $SU(N)$ gauge invariance, this is inherited by the matrix model. Indeed the gauge invariant information about an adjoint-valued matrix a is given in terms of its eigenvalues $\{a_i\}$ (here the i index labels the eigenvalue, $i = 1,\cdots N$, should not be confused with the index labelling the scalar as earlier - for the rest of this chapter we will only use i to label the eigenvalue.

The integrand of the matrix model will have the following factors, corresponding to the implementation of the saddle point method. The first is the action evaluated on the saddle. This has two contributions: one from the conformal coupling of the scalars to the curvature of S^4, and one from the auxiliary field term K_AK^A:

$$S|_{\text{saddle}} = \frac{1}{g_0^2}\frac{V(S^4)}{R^2}(2+1)\,a^2 = \frac{8\pi^2R^2}{g_0^2}\,a^2$$

where we have replaced g_{YM} with the *bare coupling* g_0, in anticipation of its renormlization.

The second factor comes from the one-loop fluctuation determinant - this is simply given by the product of determinants of the field kinetic terms, when expanded around the saddle. As usual, this is given by the product of eigenvalues of the kinetic terms - here we must include the contribution of any extra hypermultiplets we have added. In general it will have a UV divergence which must be appropriately dealt with. We can write the determinant as (we specialize to the case of $SU(N)$ gauge group with N_f massless fundamental hypermultiplets, as in our applications):

$$\begin{aligned}\mathcal{Z}_{1-\text{loop}}(a) &= \prod_{n=1}^{\infty}\left(\frac{\prod_{i<j}\left(1+\frac{R^2(a_i-a_j)^2}{n^2}\right)}{\prod_k\left(1+\frac{R^2a_k^2}{n^2}\right)^{N_f}}\right)^n \\ &= z_0\,\mathrm{e}^{R^2(2N-N_f)(\sum_{n=1}^{n_0}\frac{1}{n})(\sum_i a_i^2)}\times(\text{convergent})\end{aligned}$$

where z_0 is a (divergent) constant which can be taken into the normalization of the path integral, and we have cut off the Fourier sum at some UV scale $n_0 \equiv \Lambda R$. This formula is derived quite formally in section 4.4 of [10] using index theory, and is also discussed in [44] - though a more straightforward calculation should be possible, working directly with the action (1.15).

At large n_0 we can write

$$\sum_{n=1}^{n_0}\frac{1}{n} = \gamma + \log n_0$$

with γ Euler's constant.

When we combine this determinant factor with the classical action factor, it becomes possible to absorb the divergence by renormalizing the 't Hooft coupling $\lambda \equiv g^2N$. Explictly, we write

$$\begin{aligned}\mathrm{e}^{-\frac{8\pi^2N}{\lambda_0}R^2\sum_i a_i^2}\mathcal{Z}_{1-\text{loop}}(a) &\propto \mathrm{e}^{(-\frac{8\pi^2N}{\lambda_0}R^2+R^2(2N-N_f)(\gamma+\log\Lambda_0R))\sum_i a_i^2} \\ &= \mathrm{e}^{-\frac{8\pi^2N}{\lambda_{\text{ren}}}R^2}\end{aligned}$$

where we have defined $\lambda_{\text{ren}} \equiv \lambda$ through

$$-\frac{8\pi^2N}{\lambda_0} + (2N-N_f)(\gamma+\log\Lambda_0R) = -\frac{8\pi^2N}{\lambda_{\text{ren}}}$$

This can be expressed in terms of the dynamically generated scale $\Lambda \equiv \Lambda_0 e^{\gamma} e^{-\frac{8\pi^2 N}{\lambda_0}}$ as

$$\Lambda R = e^{-\frac{8\pi^2 N}{\lambda}\frac{1}{2N-N_f}} \tag{1.16}$$

which agrees exactly with the NSVZ beta function (1.1) when we identify the dynamical scale with the renormalization scale μ. Since localization is an exact technique, (1.16) demonstrates in the case of $\mathcal{N}=2$ theories (at least when placed on S^4) that the NSVZ formula is also exact. Since the anomalous dimensions vanish, this is a one-loop result. It is independently known that $\mathcal{N}=2$ beta functions are one-loop exact [28] - in the present case this fact follows from the saddle point limit. Specifically, when $N_f = 2N$ there is no divergence, indicating a conformal theory with exactly marginal gauge coupling. This matrix model has interesting scaling behaviour, which we examine in the next chapter.

In addition to these factors, there will be a term that arises when we go to the diagonal gauge for a. This is known as the 'Vandermonde determinant':

$$\det(\text{Vandermonde}) \equiv \prod_{i<j} (a_i - a_j)^2$$

As noted before, the supercharge used in the localization leaves invariant the Wilson line with constant scalar coupling which wraps the equator of S^4. Lastly, we must make the operator insertion of this Wilson loop into the matrix model. The insertion reads

$$\mathrm{Tr}_{\mathcal{R}}\, e^{2\pi a}$$

The exact form of this in terms of the eigenvalues $\{a_i\}$ will depend on the representation $\mathcal{R}$ of the loop - a fact that will be important later.

Putting all of these elements together, we find that the matrix model computing the expectation value of the supersymmetric equatorial Wilson loop with constant scalar coupling and in a representation $\mathcal{R}$, in $\mathcal{N}=2$ SYM on a Euclidean S^4, coupled to N_f massless hypermultiplets in the fundamental representation, is

$$\langle W_R(C)\rangle = \frac{1}{\mathrm{Vol}(SU(N))}\int \left(\prod_i \mathrm{d}a_i\right) e^{-\frac{8\pi^2}{g^2}\sum_i a_i^2} \prod_{i<j}(a_i - a_j)^2 \frac{\prod_{i<j} H^2(a_i - a_j)}{\prod_i H^{N_f}(a_i)} |\mathcal{Z}_{\text{inst}}(a)|^2\, \mathrm{Tr}_{\mathcal{R}} e^{2\pi a} \tag{1.17}$$

where

$$H(x) \equiv \prod_{n=1}^{\infty} \left[\mathrm{e}^{-\frac{x^2}{n}} \left(1 + \frac{x^2}{n^2}\right)^n \right] \tag{1.18}$$

There is one factor in (1.17) which we have not mentioned - this is the instanton factor $|\mathcal{Z}_{\text{inst}}(a)|^2$. $\mathcal{Z}_{\text{inst}}$ is Nekrasov's instanton partition function [29] for the equivariant theory on $\mathbb{R}^4$. However, in our applications we will be interested in the large-N limit in which we expect instanton contributions to be exponentially suppressed, and a careful consideration of these terms confirms this expectation [30].

The result can be extended to include N_a hypermultiplets in the adjoint representation:

$$\langle W_R(C) \rangle = \frac{1}{\mathrm{Vol}(SU(N))} \int \left(\prod_i \mathrm{d}a_i \right) \mathrm{e}^{-\frac{8\pi^2}{g^2} \sum_i a_i^2} \prod_{i<j} \frac{\prod_{i<j} (a_i - a_j)^2 \, H^2(a_i - a_j)}{\prod_{i<j} H^{2N_a}(a_i - a_j)} \cdots \tag{1.19}$$

$$\cdots \frac{1}{\prod_i H^{N_f}(a_i)} \, |\mathcal{Z}_{\text{inst}}(a)|^2 \, \mathrm{Tr}_{\mathcal{R}} \, \mathrm{e}^{2\pi a} \tag{1.20}$$

1.7 Matrix models: large N and saddle points

It is an old idea that the path integral of an $SU(N)$ gauge theory, in the limit $N \to \infty$, should be governed by a single field configuration, known as the 'master field' [31]. In general this has proved difficult to find (in a sense, AdS/CFT proves an answer at least for $\mathcal{N} = 4$ SYM with $\lambda \to \infty$). However, in 0-dimensional matrix models like that of the previous section things are more tractable: in the large-N limit, the integral over the matrix eigenvalues is dominated by a single 'saddle point' eigenvalue configuration (or possibly some discrete set of them). Since in this limit the number of eigenvalues also goes to infinity, it is convenient to work with an eigenvalue *density* instead of individual eigenvalues. This will be a real-valued function on the space in which the eigenvalues take their values. Therefore in the case of a single Hermitean matrix it is simply a function $\rho(x)$ on the reals $x \in \mathbb{R}$.

By saddle point distribution, we mean so in the standard sense of a saddle point approximation to the path integral, which becomes exact in the limit $N \to \infty$. The point is best explained by working explicitly with a simple model: the Gaussian matrix model which obtained by coupling one adjoint hypermultiplet to $\mathcal{N} = 2$ SYM in our setup, i.e. $N_f = 0$, $N_a = 1$ in (1.20). This is just $\mathcal{N} = 4$ SYM on S^4. Futhermore in turns out that the instanton factor is precisely unity in this case. Let us also remove the Wilson line insertion - then we are calculating the free energy of the theory, $F \equiv -(1/N^2)\log Z$:

$$Z = \frac{1}{\mathrm{Vol}(SU(N))} \int \left(\prod_i \mathrm{d}a_i \right) \mathrm{e}^{-\frac{8\pi^2 N}{\lambda} \sum_i a_i^2} \prod_{i<j} (a_i - a_j)^2 \tag{1.21}$$

we now take the Vandermonde factor inside the exponent, to produce an 'effective action'

$$S_{\text{eff}} = \frac{8\pi^2}{\lambda} \frac{1}{N} \sum_i a_i^2 - 2\frac{1}{N^2} \sum_{i<j} \log(a_i - a_j) \tag{1.22}$$

so that whole integral is now

$$Z = \frac{1}{\mathrm{Vol}(SU(N))} \int \left(\prod_i \mathrm{d}a_i \right) \mathrm{e}^{-N^2 S_{\text{eff}}}$$

Now $1/N^2$ is playing the role of $\hbar$ in a standard quantum mechanical saddle point method. To proceed we must find the saddle point. The condition for this is that S_{eff} is minimized with respect to all eigenvalues. Differentiating (1.22) with respect to an eigenvalue a_i we obtain

$$\frac{8\pi^2}{\lambda} a_i - \frac{1}{N} \sum_{j(\neq i)} \frac{1}{a_i - a_j} \stackrel{!}{=} 0 \tag{1.23}$$

We now must take the continuum limit, by passing to the (spectral) eigenvalue density

$$\rho(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(x - a_i)$$

which becomes a (piecewise) continuous function in the limit. Then (1.23) can be written as

$$⨍ \frac{\rho(y)}{x - y} = \frac{8\pi^2}{\lambda} x \tag{1.24}$$

where $⨍$ denotes the Cauchy principle value of the integral, since the integrand has a simple pole at $x = y$.

This integral equation is solved by an eigenvalue density which has support only in the compact intervalLarge-N eigenvalue distributions with compact support are typical in matrix models such as these. $-\sqrt{\lambda}/(2\pi) < x < +\sqrt{\lambda}/(2\pi)$, the so-called 'Wigner semi-circle distribution'

$$\rho(x) = \begin{cases} \frac{4}{\lambda}\sqrt{\lambda - (2\pi x)^2} & |x| < \frac{\sqrt{\lambda}}{2\pi} \\ 0 & |x| \geq \frac{\sqrt{\lambda}}{2\pi} \end{cases} \tag{1.25}$$

We might therefore say that this configuration of scalars is the master field for $\mathcal{N} = 4$ SYM on S^4. However, we do not know that this is the only component of the master field - since the Wilson loop is a supersymmetric observable, we are justified in expecting that other parts of the master field configuration have cancelled out when computing the path integral. Indeed, this is just what the localization procedure is doing for us: it tells us that a very particular locus of field space is all that contributes to the expectation value of this BPS object. The large-N limit, we might say, further finds the intersection of this locus with the master field.

Using this saddle point distribution, we can now calculate the expectation value of a Wilson loop in the fundamental representation:

$$\langle W_\square \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\, \rho(x)\, \mathrm{e}^{2\pi x} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \tag{1.26}$$

Expanding at large λ we find the leading piece is

$$\langle W_\square \rangle = \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{\sqrt{\lambda}}}{\lambda^{3/4}} + \cdots \tag{1.27}$$

Where $\sim$ denotes the leading term in λ. On the other hand, expanding at small λ we obtain

$$\langle W_\square \rangle = 1 + \frac{\lambda}{8} + \mathcal{O}(\lambda^2)$$

The large-λ result (1.27) is in agreement with the result (1.12) from the saddle point evaluation of the string action. The extra power of λ outside the exponential in (1.27) comes from contributions of ghosts when fixing residual conformal gauge symmetry of the 2D metric[1]. Discussion of the matrix model computations of loops in more general representations is delayed until the next chapter.

[1]The correct normalization for these modes seems to be slightly non-trivial to derive, however. See for example section 3.1 of [32] for some discussion on the issue.

Chapter 2

SQCD and large rank Wilson loops

In this chapter we consider $\mathcal{N} = 2$ SQCD, that is $\mathcal{N} = 2$ Yang-Mills coupled to N_f massless fundamental hypermultiplets, on S^4 as described in section 1.6. Most of the chapter in concerned with an example interesting for holography, namely the $N_f = 2N$ conformal theory. Finally in section 2.8 we consider the $N_f < 2N$ non-conformal case.

2.1 $N_f = 2N$ Wilson loops

Recently, Passerini and Zarembo [30] explored properties of Pestun's matrix integral for the $N_f = 2N$ $\mathcal{N} = 2$ conformal theory in the large-N limit, and deduced the behaviour of the circular Wilson loop in the fundamental representation at strong 't Hooft coupling. The focus of the present chapter is to use the large-N matrix integral for the $\mathcal{N} = 2$ SCFT to explore properties of supersymmetric Wilson loops in large representations, when the rank of the representation is of order N. Our primary motivation is to compare and contrast the results with corresponding quantities in the $\mathcal{N} = 4$ theory at strong coupling, and hence draw inferences about the possible nature of the large-N string dual, if any, of the $\mathcal{N} = 2$ SCFT.

Finding a string dual to this theory in the large-N Veneziano limit [33] at strong 't Hooft coupling, is a long standing problem. Recent proposals in this direction include [34–37]. In all cases the proposed backgrounds either contain regions of high curvature or are partly non-geometric as in [34, 37]. It is therefore interesting to ask if field theory probes such as Wilson loops can shed light on features of a putative large-N string dual. Experience with the $\mathcal{N} = 4$ theory indicates that Wilson loops in generic (large) representations can indeed act as effective probes of the dual geometry. In particular we saw in section 1.5 that Wilson loops in symmetric and antisymmetric tensor representations are computed by probe D3 and D5-branes in $AdS_5 \times S^5$ [4, 25, 38–43]. These have world-volumes $AdS_2 \times S^2 \subset AdS_5$ and $AdS_2 \times S^4$ ($S^4 \subset S^5$) respectively, and are thus sensitive to different aspects of the geometry.

In [30], a careful evaluation of the circular Wilson loop in the fundamental representation was performed at strong coupling in the $\mathcal{N} = 2$ SCFT, making use of the Pestun matrix model. Interesting, there is a non-exponential growth of its expectation value with λ at large λ[1]. This is to be contrasted with the circular Wilson loop in $\mathcal{N} = 4$ SYM, which grows exponentially at strong coupling as $\sim e^{\sqrt{\lambda}}$.

The behaviour follows from a particularly curious feature of the Pestun one-matrix model for the A_1 $\mathcal{N} = 2$ SCFT: the large-N eigenvalue distribution has an infinite support at infinite 't Hooft coupling, but most of the eigenvalues remain at finite values[2]. The limiting form of the eigenvalue distribution does not describe the fundamental Wilson loop, since the latter follows the end point of the distribution and so diverges in the limit $\lambda \to \infty$. However, in this chapter we show that the limiting distribution does describe Wilson loops in large enough tensor representations.

Specifically, Wilson loops in the antisymmetric tensor representation with rank k of order N can converge to a result that is independent of λ. In the large-N limit with $f \equiv \frac{k}{N}$ fixed, we find that the antisymmetric Wilson loops are determined by the endpoint of the eigenvalue distribution only for $f \ll \sqrt{\ln \lambda}/\lambda \ll 1$, and beyond this range approach a regular limit determined completely by the limiting eigenvalue distribution at infinite 't Hooft coupling. This behaviour strongly suggests that the internal/compact factors in the large-N string dual, probed by the corresponding D-branes, remain highly curved or non-geometric as in [34].

The situation with Wilson loops in the symmetric tensor representation of rank k turns out to be somewhat different. Their expectation values are determined by the endpoint of the eigenvalue distribution at strong coupling and essentially track the

[1]The non-exponential dependence had already been deduced via scaling arguments applied to the associated matrix model in [44].

[2]This was realized in [30]. In earlier work [44], it was noted, using scaling arguments that the eigenvalue density approaches a limiting shape.

behaviour of the fundamental Wilson loop up to a critical value of $f = f_c \sim \sqrt{\ln \lambda}/\lambda$. Beyond this point the Wilson loop experiences a non-analyticity characterized by the splitting of a single eigenvalue from the rest of the large-N distribution. A related non-analyticity was observed for the symmetric loop in $\mathcal{N} = 4$ SYM, in [42]. In that case, the position of the split eigenvalue from the large-N distribution was mapped to the position of the probe D3-brane in $AdS_5 \times S^5$ [4] which computes the symmetric Wilson loop.

The approach we use to compute the large rank Wilson loops is identical to that of [42]. We further emphasize the connection of the symmetric and anti-symmetric representations to the free Fermi and Bose gas pictures. This in turn suggests a potential tantalizing connection to the multichannel Kondo model [45–48].

The most straightforward inference we can draw from our results is that the string dual to the $\mathcal{N} = 2$ SCFT should have a weakly curved AdS_5 part which can be probed by any D-branes that compute Wilson loops in symmetric representations. The exponential growth of the latter with the 't Hooft coupling indicates that the corresponding D-branes which compute them must be semiclassical, bearing some resemblance to the situation in the $\mathcal{N} = 4$ theory. On the other hand, the behaviour of Wilson loops in the antisymmetric representation suggests that the internal or compact factor of the geometry must be highly curved. Our results may be viewed as predictions for the tensions of corresponding probe D-branes.

2.2 The matrix model

We are only interested in evaluating Wilson loops at large-N with $\lambda \equiv g^2 N$ fixed. We therefore take the continuum limit of the saddle point equation for (1.17), and specialize to the case at hand, $2N$ massless fundamental hypermultiplets:

$$\frac{8\pi^2}{\lambda} x - K(x) = \mathcal{P}\int_{-\mu}^{\mu} \mathrm{d}y\, \rho_\lambda(y) \left(\frac{1}{x-y} - K(x-y)\right), \quad x \in [-\mu, \mu] \quad (2.1)$$

$$\int_{-\mu}^{\mu} \rho_\lambda(x)\, \mathrm{d}x = 1\,, \qquad K(x) \equiv -\frac{H'(x)}{H(x)}\,.$$

where here and in the rest of the chapter we set $R = 1$ - it is simple to reinstate these factors when needed by dimension analysis. An exact solution of the saddle point equation is not (yet) known, but crucial properties of the eigenvalue density at

$\rho_\lambda(x)$ can be inferred from the behaviour of function $K(x)$ which appears both as a central force term and in the pairwise interaction of eigenvalues. As observed in [30], for small x, $K(x) \approx 2\zeta(3)x^3$ while for large x, $K(x) \to 2x\ln x$. This implies that the pairwise interaction between eigenvalues is repulsive at short separation (dominated by Vandermonde repulsion) and attractive at very large eigenvalue separation. On the other hand the central quadratic potential (attractive) dominates at short distances, but is overwhelmed by the repulsive $K(x)$ at large distances. Importantly, at large distances the one-body term $K(x)$ precisely counteracts the two-body force $K(x-y)$, so that for a large enough spread of the eigenvalue distribution, the behaviour at the endpoints is controlled by the Vandermonde repulsion and the quadratic one body potential.

We list below the main consequences [30] of these observations:

- The spectral density associated to the eigenvalues of the matrix model (2.1) above, attains a limiting form in the limit of infinite 't Hooft coupling, which is independent of the coupling,
$$\rho_\infty(x) = \frac{1}{2}\frac{1}{\cosh\left(\frac{\pi x}{2}\right)}\,. \tag{2.2}$$
The spread of the eigenvalues is infinite ($\mu \to \infty$). An important feature of the limiting form of $\rho_\infty(x)$ is that it cannot be used to yield a finite expectation value for the Wilson loop in the fundamental representation. In particular, $\langle W_\square\rangle\big|_{\lambda\to\infty} = \int_{-\infty}^{\infty} \mathrm{d}x\,\rho_\infty(x)\,\mathrm{e}^{2\pi x}$ is divergent at face value.
- Hence, the finite λ corrections to the exponential tail of the limiting distribution are crucial for determining the correct value of the Wilson loop at strong coupling. For finite (but large) λ, since the force on an eigenvalue at large distances is determined by the quadratic one-body potential and the Vandermonde repulsion, it can be argued that near its endpoints the eigenvalue distribution should smoothly interpolate between the limiting distribution and the Wigner semi-circle law,
$$\rho_\lambda(x) \simeq \frac{8\pi}{\lambda}\sqrt{\mu^2 - x^2}\,, \qquad x \sim \mu \gg 1. \tag{2.3}$$
The location of the endpoint μ can be estimated by requiring the interpolating distribution to be correctly normalized i.e., $\int_{-\mu}^{\mu} \mathrm{d}x\,\rho_\lambda(x) = 1$, assuming that the crossover between (2.2) and (2.3) occurs at $x \sim \mathcal{O}(1)$. This implies,
$$\lambda \sim \sqrt{\mu}\,\mathrm{e}^{\pi\mu/2}\,, \qquad \mu = \frac{2}{\pi}\ln\lambda + \ldots. \tag{2.4}$$

- It is then straightforward to infer the λ-dependence of $\langle W_\Box \rangle$ in the large-λ regime. The relevant integral is dominated by the endpoint of the spectral density, i.e. (2.3), so that

$$\langle W_\Box \rangle = K \frac{\lambda^3}{\ln \lambda^{3/2}} . \tag{2.5}$$

Comparison with the standard result [1] for the Wilson loop at strong 't Hooft coupling, in theories with weakly curved AdS duals [14, 32, 50–53], then suggests that the effective string tension is $T_{\text{eff}} = \frac{3}{2\pi} \ln \lambda$ in a putative string dual of the $N_f = 2N$ theory.

As we have seen, in the large-N limit, Wilson loops in large representations are computed by semiclassical D-brane probes in the (weakly curved) dual geometry [4, 25, 38–40, 42] of the form $AdS_5 \times X^5$. The antisymmetric and symmetric tensor representations are computed probe D5- and D3-branes, respectively. The former are probes of the internal geometry and their dependence on the ratio $\frac{k}{N}$ is determined essentially by the volume of the four-cycle inside X^5, wrapped by the puffed up D5-brane. On the other hand, the D3-branes are embedded completely in the AdS_5 directions with world-volume $AdS_2 \times S^2$.

It is conceivable that the behaviour of such high rank Wilson loops in the $\mathcal{N} = 2$ superconformal theory will contain some hints of a large-N string dual. It is *a priori* unclear whether such a string dual will have weak curvatures or not, but we expect that exact results from field theory may allow us to draw some inferences.

2.3 Antisymmetric representation

Given that supersymmetric Wilson loops in the $\mathcal{N} = 2$ SCFT are computed by Pestun's matrix model (1.17), Wilson loops transforming in various representations can be expressed as expectation values of appropriately symmetrized polynomials of eigenvalues of the random matrix $e^{2\pi a}$. Explicitly, we must form the the trace of $e^{2\pi a}$ in the given representation. Let us stop to make this clear. If we think of a as an adjoint-valued

[1] For large-N superconformal gauge theories with a supergravity dual (e.g. $\mathcal{N} = 4$ SYM) , the Wilson loop $W \sim \exp(\sqrt{\lambda} A)/\lambda^{3/4}$. Here A is the area of a minimal string (disk) world-sheet in AdS, whose boundary traces out the Wilson loop in the dual gauge theory. The pre-factor of $\lambda^{-3/4}$ arises from gauge-fixing on the world-sheet and depends on the number of zero modes, determined by the Euler character of the world-sheet [49].

$N \times N$ Hermitian matrix, $e^{2\pi a}$ is a matrix in the fundamental representation. Taking the trace of this, we obtain

$$\mathrm{Tr}_{\Box} e^{2\pi a} = \sum_i e^{2\pi a_i}$$

where $\{a_i\}$ are the eigenvalues of the $N \times N$ matrix a. Another way to view this is the following: by diagonalizing the matrix $e^{2\pi a}$, we have moved to a particular basis of N-dimensional vector space on which the fundamental acts: one for which the basis vectors

$$v_i \equiv \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ 0 \end{pmatrix}$$

(where the 1 is in the i'th row) satisfy $e^{2\pi a} v_i = e^{2\pi a_i} v_i$. The trace in this representation is then a sum over the states (basis vectors) of the representation of their eigenvalues.

Now let us consider the rank-k totally antisymmetric representation A_k. This is defined by taking the tensor product of k copies of the fundamental, and antisymmetrizing in all indices. Thus the independent states in A_k with definite eigenvalues are simply all possible tensor products of basis vectors of the constituent fundamentals, with no two copies of the same vector in the tensor product (since these are not independent states anymore), and with only one representative from each orbit of the permutation group $Perm(k)$ acting on the k vectors, for the same reason. So each state takes the form

$$V(A_k)_{i_1 i_2 \cdots i_k} \equiv v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq N$$

where we have used $Perm(k)$ to order the indices. Since $SU(N)$ acts in the fundamental on each of the factor spaces, this state has eigenvalue $\prod_{a=1}^{k} e^{2\pi a_{i_a}}$ with respect to $e^{2\pi a}$ (viewed now as an group element in A_k). Thus the trace in this representation, which is the sum of the eigenvalues over all states, reads

$$\mathrm{Tr}_{A_k} e^{2\pi a} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} \exp\left[2\pi(a_{i_1} + a_{i_2} + \ldots a_{i_k})\right]$$

When inserted into the path integral, this becomes

$$\frac{\langle W_{A_k}\rangle_{\mathcal{N}=2}}{\dim(A_k)} = \frac{1}{\dim(A_k)} \sum_{1\le i_1<i_2<\cdots<i_k\le N} \langle \exp\left[2\pi(a_{i_1}+a_{i_2}+\dots a_{i_k})\right]\rangle_{\mathrm{mm}}\,, \tag{2.6}$$

where the expectation value on the right hand side is within the random matrix model (1.17), and the result is normalized by the dimension of the representation, $\dim(A_k) = \binom{N}{k}$.

To evaluate this, we will follow the technique of [42], assuming that the insertion of the Wilson loop operator does not change the large-N eigenvalue distribution of the matrix model itself. This assumption is consistent in the strict large-N limit, since the matrix model exponent $\mathrm{e}^{N^2 S_{\mathrm{eff}}}$ is $\mathcal{O}(N^2)$, whilst the insertion of the rank k Wilson loop operator introduces a 'perturbation' of order N and cannot alter the large-N saddle point distribution.

Since our primary interest is in the strong coupling limit of the $\mathcal{N}=2$ gauge theory, we will compute the expectation value (2.6) simply by using the limiting form of the eigenvalue density $\rho_\infty(x)$. Despite the fact that this distribution cannot provide a finite result for the fundamental representation, we will find that large enough antisymmetric tensor representations ($\frac{k}{N}$ fixed as $N\to\infty$) *can* converge to a smooth result (at infinite λ), independent of the 't Hooft coupling λ. (Independently of this, the Wilson loops for such large rank representations will always scale exponentially with N).

The Wilson loops in the antisymmetric representation can be read off as coefficients of the characteristic polynomial for the matrix $\mathrm{e}^{2\pi a}$:

$$\langle W_{A_k}\rangle = \oint \frac{\mathrm{d}t}{2\pi i}\,\frac{1}{t^{N-k+1}}\,\langle\prod_{j=1}^{N}(t+\mathrm{e}^{2\pi a_j})\rangle_{\mathrm{mm}}\,. \tag{2.7}$$

This generating function for the antisymmetric representation has the form of a partition function for fermions in a system with N energy levels - this can be seen already in the trace over A_k, where the same eigenvalue cannot appear twice in the exponent, just as in the Pauli exclusion principal. The integral over t picks out a total number of particles k. This connection will be made more explicit later.

The large-N limit provides a particularly convenient representation of this expression as an integral over the auxiliary spectral parameter t, which can then be evaluated in a saddle point approximation,

$$\langle W_{A_k}\rangle = \oint \frac{\mathrm{d}t}{2\pi i}\,t^{k-1}\,\exp\left[N\int_{-\mu}^{\mu}\mathrm{d}x\,\rho_\lambda(x)\,\ln(1+\mathrm{e}^{-2\pi x}\,t^{-1})\right]\,. \tag{2.8}$$

Note that this integral representation takes in the eigenvalue distribution (at infinite N) as input, and can be used, in principle, to evaluate higher rank Wilson loops for any λ. It also enjoys a symmetry under the operation $k \to N-k$, which can be understood by performing the variable change $t \to t^{-1}$ and from the symmetry of $\rho_\lambda(x)$. This symmetry is equivalent to charge conjugation, so that $W_{A_k} = W_{A_{N-k}}$ as expected for sources transforming in the antisymmetric tensor representation of $SU(N)$.

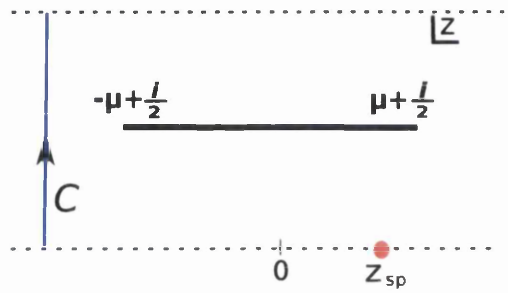


Figure 2.1: The integral along the contour C, on the cylinder yields the rank k antisymmetric Wilson loop. The contour lies to the left of the branch cut between $-\mu + \frac{i}{2}$ and $\mu + \frac{i}{2}$ where $\mu \simeq \frac{2}{\pi} \ln \lambda$. The saddle point on the first sheet lies on the real axis at $z_{\rm sp}$, well away from the branch cut.

Another feature of the formula (2.8), is that the exponent (as a function of t) has a branch cut singularity. This branch cut arises from the N zeroes of the characteristic polynomial coalescing in the large-N limit to form a continuum. The branch points are at $t = -e^{-2\pi\mu} \sim -\lambda^{-4}(\ln \lambda)^2$ and $t = -e^{2\pi\mu} \sim \lambda^4(\ln\lambda)^{-2}$. In the limit of infinitely large 't Hooft coupling the branch cut stretches along the entire negative t-axis, and we must be careful whilst considering this limit to evaluate the Wilson loop.

In the limit $k, N \to \infty$ with $\frac{k}{N}$ fixed, the t-integral can be evaluated by the method of steepest descent. The condition for the existence of a saddle-point in the t-plane is then,

$$\int_{-\mu}^{\mu} dx \, \frac{\rho_\lambda(x)}{1 + e^{2\pi x} t} - f = 0 \,, \qquad f \equiv \frac{k}{N} \,. \tag{2.9}$$

An interesting and potentially useful interpretation of the system can be obtained by adopting a different parametrization,

$$t \equiv e^{-2\pi z} \,. \tag{2.10}$$

This maps the plane to the cylinder with a branch cut along $-\mu + \frac{i}{2} \leq z \leq +\mu + \frac{i}{2}$ (figure 2.1). Along the branch cut the integrand develops an imaginary part. Given the analytic structure of the integrand, we can use this procedure as long as the location of the saddle points is away from the branch cut singularity. With the exponential parametrization, the saddle point equation acquires a nice interpretation,

$$\int_{-\mu}^{\mu} \mathrm{d}x \, \frac{\rho_\lambda(x)}{1 + \mathrm{e}^{2\pi(x-z)}} \;=\; f\,. \tag{2.11}$$

This equation gives the occupation number of fermions in the presence of a chemical potential z and with a density of states ρ_λ. Depending on the specific form of $\rho_\lambda(x)$, we may be able to infer an effective 'temperature" of the system by appropriate rescalings of the integration variable.

2.3.1 Infinite λ limit

The large-λ distribution function has the property that its endpoints run off to infinity in the limit of infinite 't Hooft coupling. We know that the fundamental Wilson loop (2.5) diverges in this limit because its expectation value is dominated by the largest eigenvalues in the vicinity of the endpoint of the eigenvalue density ρ_λ. However, we can expect large rank representations to behave differently. While individual terms in the sum (2.6) may diverge as λ is taken to infinity, in the limit of large k, N with $\frac{k}{N}$ fixed, the total number of terms in the sum grows exponentially with N. As the distribution ρ_∞ is peaked about $x = 0$, a majority of eigenvalues reside in the bulk of the distribution and their contribution to the sum can dominate the result for high rank representations. Indeed, this is precisely what occurs.

The integral representation for the antisymmetric loop at $\lambda = \infty$ explicitly reads,

$$\langle W_{A_k}\rangle\big|_{\lambda\to\infty} \;=\; i\oint_C \mathrm{d}z\,\exp\left[N\left(2\ln\left(\cosh\tfrac{\pi z}{2} + \tfrac{1}{\sqrt{2}}\right) - \left(f - \tfrac{1}{2}\right)2\pi z + \ln 4\right)\right]. \tag{2.12}$$

Despite the fact that the integrand in the original expression (2.8) for W_{A_k} is manifestly periodic under $z \to z + i$, its explicit form above in the $\lambda \to \infty$ limit, is only periodic under the shift $z \to z + 4i$. This is due to the branch cut shown in Figure(2.1) which now has an infinite extent.

The large-N saddle point equation follows directly from the above result,

$$\frac{1}{2}\sinh\tfrac{\pi z}{2}\left(\cosh\tfrac{\pi z}{2} + \tfrac{1}{\sqrt{2}}\right)^{-1} \;=\; f - \frac{1}{2}\,. \tag{2.13}$$

This equation has distinct roots with $\operatorname{Im} z = 0$ and $\operatorname{Im} z = 2i$. For the moment, we focus our attention only on the real root. The complex root lies on a different sheet, due to the branch cut discussed above. We find one saddle point on the real axis satisfying,

$$\coth\left(\tfrac{\pi}{2}\, z_{\text{sp}}\right) = \frac{1 - \sqrt{\frac{1}{2} - \left(f - \frac{1}{2}\right)^2}}{f - \frac{1}{2}}. \tag{2.14}$$

As a function of f, in the domain $0 \leq f \leq 1$, z_{sp} takes on all values on the real axis. For small f and f approaching unity, the saddle point runs of to infinity,

$$\begin{aligned} z_{\text{sp}} &= \tfrac{2}{\pi} \ln f + \ldots\,, & f \ll 1\,, \\ &= -\tfrac{2}{\pi} \ln(1-f) + \ldots\,, & (1-f) \ll 1\,. \end{aligned} \tag{2.15}$$

A nice property of this solution as a function of f, is that

$$z_{\text{sp}}(f) = -z_{\text{sp}}(1-f)\,. \tag{2.16}$$

This reflects the symmetry of the integral representation for the antisymmetric Wilson loop, under $z \to -z$ accompanied by the replacement $f \to 1-f$. In the free fermion interpretation, z_{sp} acts as a real chemical potential. Since the distribution $\rho_\infty(x)$ is independent of λ, the effective temperature for the Fermi-Dirac distribution in (2.11) is $\mathcal{O}(1)$.

We now proceed to evaluate the Wilson loop itself. After deforming the contour to pick up the saddle point contribution (formally, before taking the $\lambda \to \infty$ limit) from the first sheet, we have

$$\langle W_{A_k}\rangle\big|_{\lambda\to\infty} \approx \exp\left[N\left(2\ln\left[\frac{\sinh\left(\frac{\pi}{2} z_{\text{sp}}\right)}{f - \frac{1}{2}}\right] - \left(f - \tfrac{1}{2}\right) 2\pi z_{\text{sp}}\right)\right]. \tag{2.17}$$

The explicit expression is not particularly illuminating, but the expansion of the exponent in a power series in $f = \frac{k}{N}$ is interesting:

$$\boxed{\ln\langle W_{A_k}\rangle\big|_{\lambda\to\infty} = N\left[-4\frac{k}{N}\ln\left(\frac{\sqrt{2}}{e}\frac{k}{N}\right) - \frac{8}{3}\left(\frac{k}{N}\right)^3 + 4\left(\frac{k}{N}\right)^4 \ldots\right]} \qquad \frac{k}{N} \ll 1\,. \tag{2.18}$$

The most notable features of this formula[1] are the non-analytic (logarithmic) dependence on f for small f, the complete absence of any dependence on the 't Hooft

[1]Note that our definition of W_{A_k}, (2.6), does not include the normalization factor given by the dimension of the representation.

coupling λ and that beyond the leading order, subsequent terms are organized in a power series in $\frac{1}{N}$ (as opposed to $\frac{1}{N^2}$, for instance). The logarithmic behaviour is an immediate consequence of (2.17) and that $z_{\rm sp} \sim \ln f$ for small f.

2.3.2 Large but finite λ

For any finite value of the coupling constant, the eigenvalue distribution has a finite extent [30], and the exponential tail of $\rho_\lambda(x)$ smoothly matches on to a Wigner semi-circle distribution which vanishes at $|x| = \mu \approx \frac{2}{\pi}\ln\lambda + \ldots$. In the fixed k, large-N limit, we expect that the value for the (exponent of) the antisymmetric Wilson loop is k times that of the fundamental loop (from large-N factorization). The result for the fundamental loop (2.5) depends on λ, whereas we have seen that for large enough representations with $f = \frac{k}{N}$ fixed, the observables have a λ-independent limit at large λ. We will now see how these two results can be reconciled.

To understand that there must be a qualitative change in the behaviour, we do not need the explicit form of ρ_λ. All we require is that the eigenvalue distribution has a finite extent for finite 't Hooft coupling. In particular we know that the distribution is non-zero only in the domain $-\mu < x < \mu$. The relevant question is whether the solution to (2.9) is sensitive to this finite extent of the distribution. When f is taken to be sufficiently small, we can find a saddle point with $\mathrm{e}^{-2\pi z} \gg \mathrm{e}^{2\pi\mu}$, by expanding (2.9) in powers of $\mathrm{e}^{2\pi z}$ and keeping the leading term

$$\mathrm{e}^{-2\pi z} \simeq \frac{1}{f}\int_{-\mu}^{\mu} \mathrm{d}x\, \rho_\lambda(x)\, \mathrm{e}^{-2\pi x} = \frac{1}{f}\langle W_\Box\rangle\,, \qquad \mathrm{e}^{-2\pi z} \gg \mathrm{e}^{2\pi\mu}\,. \tag{2.19}$$

We already know that $\langle W_\Box\rangle \sim \mathrm{e}^{3\pi\mu/2} \sim \lambda^3/(\ln\lambda)^{3/2}$ at strong coupling, and therefore the above solution applies for small f such that

$$f \ll \mathrm{e}^{-\pi\mu/2} \sim \lambda^{-1}\sqrt{\ln\lambda} \ll 1\,. \tag{2.20}$$

Substituting the small f saddle point into the integral representation for the antisymmetric Wilson loop, we obtain

$$\frac{\langle W_{A_k}\rangle}{\dim(A_k)} \quad\to\quad \frac{k!(N-k)!}{N!}\exp\left(N\left[f + f\ln\left(f^{-1}\langle W_\Box\rangle\right)\right]\right) = \langle W_\Box\rangle^k\,. \tag{2.21}$$

Note that the overall normalization given by the dimension of the representation is precisely cancelled by contributions to the exponent at the saddle point. Therefore, explicitly, we have

$$\langle W_{A_k}\rangle \sim \frac{\lambda^{3k}}{(\ln\lambda)^{3k/2}}\,, \qquad \text{for} \quad f \ll \mathrm{e}^{-\pi\mu/2}\,. \tag{2.22}$$

Hence, the antisymmetric Wilson loop at strong coupling exhibits a crossover from λ-dependent behaviour (2.22) for $\frac{k}{N} \ll \lambda^{-1}\sqrt{\ln\lambda}$ to a λ-*independent* limit (2.18) for parametricaly larger $f = \frac{k}{N}$. We do not know if the cross-over is actually smooth, but given that the saddle-point $z_{\rm sp}$ (see figure(2.2)) is located well away from the branch points, we do not expect non-analyticities in the associated configuration. Furthermore, we have not seen evidence of more than one saddle point, so a first order transition due to competition between two or more configurations appears unlikely. Settling the actual nature of the crossover behaviour described above will require a more detailed analytical or numerical understanding of the finite-λ eigenvalue distribution.

2.4 Comparison with $\mathcal{N} = 4$ SYM

A comparison of the result above with the corresponding one for $\mathcal{N} = 4$ SYM is quite useful. We showed in section 1.7 that circular Wilson loops in $\mathcal{N} = 4$ SYM are computed by the Gaussian matrix model, whose saddle point density is the Wigner semi-circle law (1.25). Using this eigenvalue distribution and upon rescaling variables in (2.11), $\tilde{x} \equiv \frac{\sqrt{\lambda}}{2\pi}\,x$ and $\tilde{z} \equiv \frac{\sqrt{\lambda}}{2\pi}\,z$, the saddle point value of the rank k antisymmetric Wilson loop is determined by the condition,

$$\frac{2}{\pi}\int_{-1}^{1} \mathrm{d}\tilde{x}\, \frac{\sqrt{1-\tilde{x}^2}}{1+\mathrm{e}^{\sqrt{\lambda}(\tilde{x}-\tilde{z})}} = f\,. \tag{2.23}$$

The crucial difference with respect to the $\mathcal{N} = 2$ theory studied previously, is the explicit dependence on the 't Hooft coupling of the $\mathcal{N} = 4$ theory, which now plays the role of the temperature of the Fermi-Dirac distribution.

The limit of infinitely strong coupling ($\lambda \to \infty$) can be interpreted as a 'zero temperature' limit for free fermions and therefore all Fermi levels $\tilde{x}$ below the chemical potential $\tilde{z}$ are 'occupied', whilst all $\tilde{x} > \tilde{z}$ remain 'empty'. Hence, the saddle point $\tilde{z}_{\rm sp}$ is determined by setting the total number of fermions (eigenvalues) in the ground state to k

$$\frac{2}{\pi}\int_{-1}^{\tilde{z}_{\rm sp}} \mathrm{d}y\,\sqrt{1-\tilde{x}^2} = \frac{k}{N}\,. \tag{2.24}$$

This leads to the known result, quoted at the end of chapter 1 [4, 39, 42],

$$\langle W_{A_k}\rangle_{\mathcal{N}=4}\big|_{\lambda\to\infty} = \exp\left(N\,\frac{2\sqrt{\lambda}}{3\pi}\sin^3\theta_k\right)\,, \qquad \frac{1}{\pi}(\theta_k - \sin\theta_k\cos\theta_k) = \frac{k}{N}\,. \tag{2.25}$$

This result is only valid when $\lambda\to\infty$. For any finite λ, our free fermion interpretation above shows that there will be corrections due to occupation of all levels above the chemical potential. It would be interesting to look at the detailed form of the corrections to the infinite coupling limit. The leading corrections include powers of λ^{-1} and exponentially suppressed terms $\sim \mathrm{e}^{-\sqrt{\lambda}(1-|\tilde{z}_{\mathrm{sp}}|)}$ which are suggestive of world-sheet instanton corrections to the D5-brane saddle point. In the free Fermi picture, these corrections can be thought of as contributions from excited levels at a finite temperature of order $\lambda^{-1/2}$.

The salient features of the formula (2.25) are the dependences on N and $\sqrt{\lambda}$. The exponential growth with N is due to the large k limit with $k\sim\mathcal{O}(N)$, while the factor of $\sqrt{\lambda}$ is a consequence of the large 't Hooft coupling. In the IIB string dual description on $AdS_5\times S^5$, the dependence on N and $\sqrt{\lambda}$ arises straightforwardly from the tension of a semiclassical D5-brane wrapping a flux supported $S^4\subset S^5$, which computes the antisymmetric Wilson loop [4, 38–40]. The D5-brane tension $T_{\mathrm{D5}} = 1/(2\pi g_s\alpha'^3)$ expressed as a function of the 't Hooft coupling and AdS radius R_{AdS} becomes $N\sqrt{\lambda}/8\pi^4R^6_{\mathrm{AdS}}$. Since all the curvature scales are set by R_{AdS}, the scaling of the Wilson loop is completely determined. The angular variable θ_k introduced above determines a specific latitude in S^5 and the supergravity result precisely matches (2.25).

The result for the $\mathcal{N}=2$ SCFT suggests that the antisymmetric Wilson loop for large k, in a putative large-N string dual, should indeed be computed by a D-brane since its action is $\mathcal{O}(N)\sim g_s^{-1}$, where g_s is the string coupling. However, the absence of any dependence on the 't Hooft parameter λ also indicates that this D-brane must probe (internal) portions of the geometry with string scale curvatures, since we saw in section 1.5.3 that the λ dependence in the exponent measures the size of the of brane in string units. For sufficiently small $\frac{k}{N}$ the antisymmetric loop is k times the fundamental Wilson loop, which, due to its growth with λ is likely computed by a string in AdS_5 with a large effective tension $T_{\mathrm{eff}} = \frac{3}{2\pi}\ln\lambda$. The transition/crossover from this semiclassical picture to a less familiar situation could be driven by $k\sim\mathcal{O}(N)$ strings “puffing up” into a D-brane probing highly curved or non-geometric portions of the dual string theory.

2.5 Symmetric Representation

The states of the rank-k totally symmetric representation S_k are again formed as tensor products of k fundamentals. The only difference is that now we have extra states in which some of the vectors in the tensor product are the same:

$$V(S_k)_{i_1 i_2 \cdots i_k} \equiv v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} \qquad i_1 \leq i_2 \leq \cdots \leq i_k \qquad .$$

again with eigenvalue $\prod_{a=1}^{k} \mathrm{e}^{2\pi a_{i_a}}$. When inserted into the matrix model this gives

$$\frac{\langle W_{S_k} \rangle_{\mathcal{N}=2}}{\dim(S_k)} = \frac{k!(N-1)!}{(N+k-1)!} \sum_{1 \leq i_1 \leq i_2 < \ldots \leq i_k \leq N} \langle \exp\left[2\pi(a_{i_1} + a_{i_2} + \ldots a_{i_k})\right] \rangle_{\mathrm{mm}} \,. \qquad (2.26)$$

The symmetric tensor representations can be obtained by considering a generating function which is the inverse of the characteristic polynomial,

$$\begin{aligned} \langle W_{S_k} \rangle &= \oint \frac{\mathrm{d}t}{2\pi i}\, t^{k-1} \left\langle \frac{1}{\prod_{i=1}^{N}(1 - t^{-1}\,\mathrm{e}^{2\pi a_i})} \right\rangle_{\mathrm{mm}}, \qquad (2.27) \\ &\to i \oint_C \mathrm{d}z\, \exp\left[-N \int_{-\mu}^{\mu} \mathrm{d}x\, \rho_\lambda(x)\, \ln(1 - \mathrm{e}^{-2\pi(x-z)}) - 2\pi k z\right] . \end{aligned}$$

As noted above, the main difference between this and the antisymmetric tensor representation is that now the eigenvalues a_i appearing in each term of the polynomial need not all be distinct - indeed the generating function now takes the form of a partition function for a system of bosons. In fact, we may separate out the terms into two categories: those with $i_1 \neq i_2 \neq \ldots \neq i_k$, and those where some of the i_ℓ coincide. The contribution from the former is identical to the antisymmetric representation; the latter includes terms where most eigenvalues coincide so that their behaviour is like a multiply wound loop. In fact we will see below that the integral representation is dominated by two distinct saddle points, related precisely to the two categories of terms in the symmetrized polynomial. At strong coupling the saddle point related to the multiply wound loop grows exponentially with the 't Hooft coupling and dominates the symmetric loop.

The integral representation above which leads to the large N limit, involves a function with a branch cut along the real axis $-\mu \leq z \leq \mu$. Therefore, we need to first determine whether putative saddle points that contribute to the integral, lie in the vicinity of the branch cut on the real axis. Now the large-N saddle point equation can be interpreted as fixing the number of bosons at a chemical potential z, with a density of states given by $\rho_\lambda(x)$,

$$\int_{-\mu}^{\mu} \mathrm{d}x \, \frac{\rho_\lambda(x)}{\mathrm{e}^{2\pi(x-z)}-1} \; = \; f \,. \tag{2.28}$$

Formally, the equation for the symmetric representation can be obtained from (2.9) after the replacement $f \to -f$ and $z \to z + \frac{i}{2}$. However, now the analogy with the Bose-Einstein distribution implies that the system could display a phase transition analogous to Bose-Einstein condensation as the chemical potential is dialled from low to high values.

2.5.1 Large λ and a non-analyticity

Small f solution: We begin by locating the solution to (2.28), with the finite λ distribution function. It is clear that the equation is likely to have solutions with $z < -\mu$, for then the integrand is positive definite. A quick check will now confirm this expectation. Assuming that the equation is solved by some $\mathrm{e}^{2\pi z} \ll \mathrm{e}^{-2\pi\mu}$, and expanding (2.28) to the lowest order in $\mathrm{e}^{2\pi z}$, we find a saddle point given by

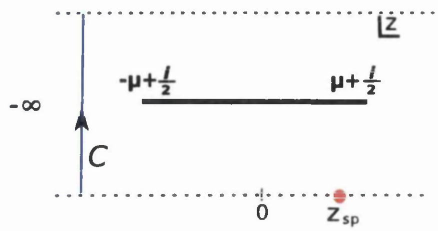
$$\exp(-2\pi z_{\rm sp}) \approx \frac{2\mathrm{e}^{3\pi\mu/2}}{3\pi f} \quad \Longrightarrow \langle W_{S_k}\rangle \propto \frac{\lambda^{3k}}{(\ln\lambda)^{3/2}}\,, \qquad f \ll \lambda^{-1}\sqrt{\ln\lambda} \ll 1\,. \tag{2.29}$$

This is identical to the antisymmetric loop in the same limit, and is consistent with the view that in the fixed k large N limit, the two should be equivalent to k coincident Wilson loops in the fundamental representation. In fact, we can be more precise. Following the same arguments that led to (2.21), we find

$$\frac{\langle W_{S_k}\rangle}{\dim(S_k)} \quad \to \quad \langle W_\Box \mathrm{e}^k \qquad \text{for} \quad f \ll \lambda^{-1}\sqrt{\ln\lambda} \ll 1\,. \tag{2.30}$$

What is now interesting in the case of the symmetric representation is that for a given (large) λ, as f is increased smoothly, the saddle point moves to larger values of z, towards the branch-point at $z = -\mu$. Beyond this, the nature of the saddle point must change as it moves through the branch cut joining $z = -\mu$ and $z = +\mu$. Let us now ascertain the critical value $f = f_c$ at which this occurs. A useful way to proceed is to first formally expand (2.9) as a power series in $\mathrm{e}^{2\pi z}$, assuming $z < -\mu$:

$$\sum_{n=1}^{\infty} \mathrm{e}^{2n\pi z} \int_{-\mu}^{\mu} \mathrm{d}x \, \rho_\lambda(x)\, \mathrm{e}^{-2\pi n x} \; = \; f \,. \tag{2.31}$$



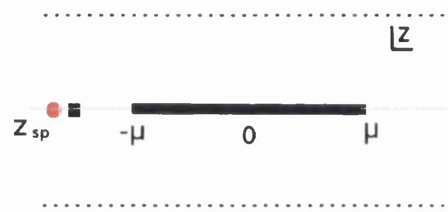


Figure 2.2: **Left**: The branch cut singularity associated to the integral representation of the symmetric Wilson loop lies on the real axis in the z-plane. For f less than a critical value f_c the saddle point $z_{\rm sp}$ also lies on the real axis. **Right**: When $f > f_c$ the saddle crosses the branch point. Beyond, a qualitatively new saddle point dominates the integral: a single eigenvalue well-separated from the continuum, reminiscent of Bose condensation. A similar configuration in $\mathcal{N} = 4$ SYM was related by [4] to the position of a probe D3-brane in $AdS_5 \times S^5$.

The coefficients in this expansion are the VEVs of multiply wound loops in the fundamental representation. Following the logic of the arguments presented in [30], these should be determined completely by the endpoint of the eigenvalue distribution at strong coupling i.e. the endpoint of the Wigner semi-circle law (2.3). At strong coupling ($\mu \gg 1$), we have

$$\int_{-\mu}^{\mu} \mathrm{d}x\, \rho_\lambda(x)\, \mathrm{e}^{-2\pi n x} \simeq R\, \frac{\sqrt{\mu}}{\lambda}\, \frac{\mathrm{e}^{2n\pi\mu}}{n^{3/2}}\,, \tag{2.32}$$

where the right hand side follows completely from the square root behaviour near the endpoints of the eigenvalue distribution. This assumption yields $R = 2$; a more careful estimate of R performed in [30], gave $R = 2.18$ (for $n = 1$). We note that in principle R could depend on n, but we take this dependence to be weak. With increasing n, the integral is more sharply peaked near the endpoint at $x = \mu$ and we expect that taking $R = 2$ becomes a better approximation. The strong coupling saddle point equation can then be rewritten (assuming $\mathrm{e}^{2\pi z} < \mathrm{e}^{-2\pi\mu}$),

$$R\, \frac{\sqrt{\mu}}{\lambda}\, \mathrm{Li}_{\frac{3}{2}}\left(\mathrm{e}^{2\pi(\mu+z)}\right) = f\,. \tag{2.33}$$

The critical value of f for which the saddle hits the branch point at $z = -\mu$ is,

$$f_c \simeq R\sqrt{\tfrac{2}{\pi}}\,\zeta\left(\tfrac{3}{2}\right)\lambda^{-1}\sqrt{\ln\lambda}\,. \tag{2.34}$$

Pushing further the analogy with the boson gas, this phenomenon is reminiscent of the onset of Bose-Einstein condensation, as the chemical potential z is dialled towards the lowest energy level (eigenvalue) at $x = -\mu$. At this point one should treat the lowest level (eigenvalue) and its occupation number separately, so that it is 'split' from the higher levels. Motivated by this line of thinking we find a result quite similar to that of [4] for the multiply wound Wilson loop in $\mathcal{N} = 4$ SYM.

Large f solution ($f > f_c$): Following the intuition gained from the free boson gas picture, we allow for the possibility that for $f > f_c$, the occupation number associated to the lowest eigenvalue a_1, at the edge of the distribution, is macroscopic

$$n_1 \equiv \frac{1}{\mathrm{e}^{2\pi(a_1 - z)} - 1} \sim \mathcal{O}(N)\,. \tag{2.35}$$

This assumption alters the saddle point equations obtained from the matrix model (2.1) for this eigenvalue alone, while leaving unaltered the large-N distribution of the remaining $N-1$ eigenvalues.

To extract the behaviour of the symmetric Wilson loop for $f > f_c$, we turn to the discrete (finite N) version of the saddle point of the matrix integral (2.1), but in the presence of an insertion of the generating function for symmetric loops, i.e. (2.27). In the large-N limit (assuming $n_1 \sim \mathcal{O}(N)$), we arrive at two saddle point equations upon varying with respect to a_1 and with respect to t:

$$\frac{8\pi^2}{\lambda}\,a_1 - K(a_1) - \int_{-\mu}^{\mu} \mathrm{d}x\,\rho_\lambda(x)\left(\frac{1}{a_1 - x} - K(a_1 - x)\right) = -\pi\frac{n_1}{N}\,, \tag{2.36}$$

$$\frac{n_1}{N} + \int_{-\mu}^{\mu} \mathrm{d}x \frac{\rho_\lambda(x)}{\mathrm{e}^{2\pi(x-z)} - 1} = f\,. \tag{2.37}$$

The rest of the $(N-1)$ eigenvalues continue to be governed by the unperturbed distribution $\rho_\lambda(x)$. To establish the exact form of the split eigenvalue solution for $f \gtrsim f_c$, just above the transition, we would need the explicit form of $\rho_\lambda(x)$ for finite, large λ. However, the exact solution of the large-N equation (2.1) is unknown at finite λ. Despite this, it is easy to infer the nature of this saddle point for large enough λ or $\frac{k}{N}$, as we now see.

We look for a solution to the above set of equations by assuming that $z \ll -\mu$. We may then safely ignore the exponentially small contribution from the continuum in (2.37), so that

$$\frac{n_1}{N} \approx f \implies a_1 \approx z + \frac{2\pi}{k} \ll -\mu\,. \tag{2.38}$$

Using the asymptotic form of $K(x)$ we find that the force on an eigenvalue, a_1, at large distances from the continuum distribution, is only due to the harmonic central potential. This must be balanced by the constant force on the right hand side of (a), yielding,

$$a_1 \approx -\frac{\lambda}{8\pi}\frac{k}{N}\,. \tag{2.39}$$

This result is valid as long as,

$$a_1 \ll -\mu, \qquad \text{or} \qquad \frac{\lambda}{8\pi}\frac{k}{N} \gg \frac{2}{\pi}\ln\lambda\,, \tag{2.40}$$

which is easily satisfied at strong coupling if we take $f = \frac{k}{N} \sim \mathcal{O}(1)$. Notice that this implies a lower bound on f for fixed large λ, which is safely above f_c and so the solution above describes the Wilson loop in the correct 'phase'.

Substituting our solution into (2.27), and taking care to include the contribution from the quadratic potential term in (2.1) we find

$$W_S^{(1)} \approx \exp\left[2N\left(\sqrt{\lambda}\frac{k}{4N}\right)^2\right]\,. \tag{2.41}$$

This result has corrections to its exponent, scaling as $\sim \mathrm{e}^{-\frac{\lambda k}{4N}}$. We have introduced a superscript to identify the contribution to the Wilson loop from this saddle point. Below we will encounter another strong coupling saddle point.

Interestingly, the entire analysis above could have been adapted to calculate the symmetric Wilson loop in $\mathcal{N} = 4$ SYM, yielding exactly the same result as (2.41), provided $\frac{k}{N}$ is taken to be large enough. Indeed, the known exact formula for that case (both from the probe D3-brane calculation [25] and the Gaussian matrix model [4, 41, 42]) reduces to (2.41) when $\sqrt{\lambda}\frac{k}{N} \gg 1$. These observations indicate that symmetric Wilson loops may be computed by a semiclassical object (D-brane) in the large-N string dual of the $\mathcal{N} = 2$ SCFT and that, in particular, the AdS_5 portion of the geometry probed by it may not be highly curved.

2.5.2 A second strong coupling saddle point

We found above that the saddle point contribution for the circular Wilson loop in the symmetric tensor representation grows exponentially with λ (and with N) at large λ. This is in stark contrast to the antisymmetric loop. However, it turns out that there is another saddle point which contributes to the expectation value of the symmetric Wilson loop and is λ-independent at infinite λ. The quickest way to see this is to note that the integrand in (2.27) and its associated saddle point equation can be obtained, up to an overall sign, from those for the antisymmetric representation after the replacements: $z \to z + \frac{i}{2}$ and $f \to -f$. At this new saddle point

$$W_S^{(2)}|_{\lambda\to\infty} = \tag{2.42}$$
$$\exp\left[-N\left(2\ln\left(\cosh\tfrac{\pi}{2}(z_{\rm sp}+\tfrac{i}{2})+\tfrac{1}{\sqrt{2}}\right)+\left(f+\tfrac{1}{2}\right)2\pi(z_{\rm sp}+\tfrac{i}{2})+\ln 4\right)\right],$$

$$\coth\left(\tfrac{\pi}{2}(z_{\rm sp}+\tfrac{i}{2})\right) = \frac{\sqrt{\frac{1}{2}-\left(f+\frac{1}{2}\right)^2}-1}{f+\frac{1}{2}}.$$

Expanding the contribution for small $f = \frac{k}{N}$,

$$W_S^{(2)}|_{\lambda\to\infty} = \exp N\left[-4\frac{k}{N}\ln\left(\frac{\sqrt{2}}{\mathrm{e}}\frac{k}{N}\right)-\frac{40}{3}\left(\frac{k}{N}\right)^3-36\left(\frac{k}{N}\right)^4+\ldots\right]. \tag{2.43}$$

The full symmetric Wilson loop at large N, is given by the sum of the contributions from the two saddle points,

$$\langle W_{S_k}\rangle = W_S^{(1)} + W_S^{(2)}. \tag{2.44}$$

Hence the symmetric Wilson loops can exhibit yet another type of non-analyticity, namely, a first order phase transition at large N due to a competition between these two configurations. However, for fixed f and large λ, it appears that $W_S^{(1)}$ will always dominate over the second saddle point, which remains exponentially suppressed.

The existence of two saddle points for the symmetric representation can be intuitively understood from the symmetrized polynomial representation (2.26). One configuration is dominated by the terms which compute the multiply wound Wilson loop $(\mathrm{Tr}e^{2\pi k a})$ and these have $i_1 = i_2 = \ldots i_k$, whilst the second type of configuration is dominated by the terms with all or most eigenvalues being distinct and is therefore similar to the antisymmetric loop. A similar transition has also been noted in $\mathcal{N}=4$ SYM [41, 42].

2.6 Discussion

The matrix model described in section 1.6 provides a powerful tool for extracting predictions for a large class of observables in $\mathcal{N} = 2$ SCFTs. It may also serve as a potential window into large-N string duals of SCFTs at strong coupling. The analysis in this chapter presents us with one class of observables in the $N_f = 2N$ theory that could shed light on the corresponding large-N dual. Below we discuss some related questions and directions for future study.

Relation to Kondo models : Wilson loops in different representations can be viewed as impurity spins [40] coupled to an ambient theory which, in our case, is an $\mathcal{N} = 2$ SCFT. These may be regarded as supersymmetric versions of the Kondo model (see e.g. [45, 46]). An antisymmetric Wilson loop computes the action of a fermionic impurity interacting with the SCFT. There is at least one SCFT namely, $\mathcal{N} = 4$ SYM, for which the circular Wilson loop in the ambient theory at zero temperature, has been related to the impurity model (or Polyakov loop) at finite temperature [1] [47, 48].

It is therefore not unreasonable to expect a connection between the Wilson loops we have computed and certain large-N Kondo models. In this context it would be interesting to understand the physical origin of the non-zero effective "temperature" we see in our zero temperature calculations, both for $\mathcal{N} = 4$ theory (2.23) and for the $\mathcal{N} = 2$ SCFT (2.11). In the former case the effective "temperature" associated to the fermion impurity action scales as $1/\sqrt{\lambda}$ and parametrizes the stringy corrections to the D5-brane action computing the antisymmetric Wilson loop. For the $\mathcal{N} = 2$ SCFT at infinite 't Hooft coupling the corresponding equation (2.11) which fixes the occupation number of the impurity fermions has no free parameters and appears to be fixed at an effective temperature of $\mathcal{O}(1)$. The resulting action for the antisymmetric Wilson loop (2.18) bears a striking resemblance to the impurity entropy for the large-N multichannel Kondo model of [45] (setting $K = N$ in that model). In particular, both share the same logarithmic dependence on the fermion occupation number k/N.

[1]The origin of this relation has been explained in [47] at strong coupling. In an SCFT the size of a circular loop, or the temperature, can always be rescaled to unity. It is not *a priori* clear that the circular loop at zero temperature should be related to the Polyakov loop at finite temperature since one is supersymmetric while the other has anti-periodic boundary conditions for ambient fermions around the thermal circle.

Motivated by this resemblance between the two systems, we make a purely empirical observation which appears to follow solely from the strong coupling eigenvalue density $\rho_\infty(x) = 1/(2\cosh\frac{\pi x}{2})$. Suppose that we introduce a fictitious "temperature" β into our large-N equations for the antisymmetric loop so the equation for the fermion occupation number (2.11) reads,

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{2\cosh\frac{\pi x}{2}} \, \frac{1}{1+\mathrm{e}^{2\pi\beta(x-z)}} = f. \tag{2.45}$$

In the zero temperature limit $\beta \to \infty$, where only the states below the chemical potential z are occupied, we obtain that the saddle point (chemical potential) and impurity action are

$$z = \tfrac{2}{\pi} \ln \tan\left(\tfrac{\pi f}{2}\right), \quad \ln\langle W_{A_k}\rangle\big|_{\beta\to\infty} = \frac{8}{\pi}\beta N \left[\mathrm{Im}\,\mathrm{Li}_2\left(i\tan\tfrac{\pi f}{2}\right) - \tfrac{f\pi}{2}\ln\tan\left(\tfrac{\pi f}{2}\right)\right], \tag{2.46}$$

When expanded out for small $f = \frac{k}{N}$ this yields,

$$\ln\langle W_{A_k}\rangle\big|_{\beta\to\infty} = 4\beta N\left[1 - \ln\left(\frac{\pi}{2}\frac{k}{N}\right) - \frac{\pi^2}{36}\left(\frac{k}{N}\right)^3 + \ldots\right], \tag{2.47}$$

precisely matching the impurity entropy for the large-N multichannel Kondo model [45, 48]. The physical significance of this is unclear at the moment as we do not really have a means to introduce a tunable effective "temperature" in the matrix model for the $N_f = 2N$ theory. In the $\mathcal{N} = 4$ theory, as explained earlier, the 't Hooft coupling appears to play this role. It is possible that studying Wilson loops in more general $\mathcal{N} = 2$ quiver SCFT's (such as the $SU(N) \times SU(N)$ theory) might shed light on this connection, if any, to Kondo models.

Aspects of large-N duals: The study of supersymmetric Wilson loops in even larger representations, with ranks of order N^2, proved to be remarkably fruitful in the context of the $\mathcal{N} = 4$ theory. The relation between the matrix model picture for such operators in terms of back-reacted eigenvalue distributions [4, 54], and their gravity duals, has been understood via explicit construction of half-BPS type IIB "bubbling geometries" [22, 23]. These geometries incorporate the complete back-reaction of the D3 and D5-branes dual to the large Wilson loop operators. It would be interesting to pursue the study of these large rank ($\mathcal{O}(N^2)$) Wilson loops in the $N_f = 2N$ theory using Pestun's matrix model. It is possible that the large-N behaviour of these operators will provide us with further clues about the large-N string dual of this theory. The first steps towards this computation are sketched in section 2.7.

Finally, it has been argued recently [34] that the string dual to the $\mathcal{N}=2$ SCFT in the Veneziano limit is a non-critical string background with seven geometric dimensions. It would be extremely interesting to know if the results found for large Wilson loops in this chapter could be naturally interpreted as expected features of branes in such non-critical backgrounds.

2.7 $O(N^2)$ Young Tableaux in $\mathcal{N}=2$ SCQCD

In this section we derive a matrix model for 'even larger' representations of $SU(N)$, i.e. those whose Young tableaux are made up of $\mathcal{O}(N^2)$ boxes, by restating a derivation appearing in [55].

There is a well known formula for the character of an $SU(N)$ representation, expressed in terms of its Young tableau:

$$\mathrm{Tr}_R \mathrm{e}^{2\pi a} = \frac{\det(\mathrm{e}^{2\pi(N+R_i-i)a_j})}{\det(\mathrm{e}^{2\pi(N-i)a_j})} \tag{2.48}$$

The following formulae hold:

$$\det A_{N\times N} = \sum_{\sigma\in S_N} \mathrm{sgn}(\sigma)\prod_i A_{i\,\sigma(i)} \tag{2.49}$$

$$\det(\mathrm{e}^{2\pi(N-i)a_j}) = \prod_{i<j}(\mathrm{e}^{2\pi a_i} - \mathrm{e}^{2\pi a_j})\,. \tag{2.50}$$

We want to insert this into the path integral to find the matrix model for the Wilson loop in representation R:

$$\begin{aligned}\langle W_R\rangle &= \frac{1}{Z}\int[\mathrm{d}a]\cdots \mathrm{Tr}_R\mathrm{e}^{2\pi a} \qquad (2.51)\\ &= \frac{1}{Z}\int[\mathrm{d}a]\cdots \sum_{\sigma\in S_N}\mathrm{sgn}(\sigma)\prod_i \mathrm{e}^{2\pi(N+R_i-i)a_{\sigma(i)}}\left(\prod_{i<j}(\mathrm{e}^{2\pi a_i}-\mathrm{e}^{2\pi a_j})\right)^{-1}.\end{aligned}$$

Manipulating the sum over the symmetric group:

$$\langle W_R\rangle = \frac{1}{Z}\int[\mathrm{d}a]\cdots\sum_{\sigma\in S_N}\frac{\prod_i \mathrm{e}^{2\pi(N+R_i-i)a_{\sigma(i)}}}{\prod_{i<j}(\mathrm{e}^{2\pi a_{\sigma(i)}}-\mathrm{e}^{2\pi a_{\sigma(j)}})} \tag{2.52}$$

We now use the fact that the a_i's are 'dummy variables' within the integral, and so we can relabel them at will. This leads to

$$\langle W_R \rangle = \frac{1}{Z} \int [\mathrm{d}a] \cdots N! \frac{\prod_i \mathrm{e}^{2\pi(N+R_i-i)a_i}}{\prod_{i<j}(\mathrm{e}^{2\pi a_i} - \mathrm{e}^{2\pi a_j})} \tag{2.53}$$

Now we label the eigenvalues. We use the notation of e.g. [55]. The eigenvalues split into groups labelled by the blocks of the Young tableau:

$$\begin{array}{l} u_1^{(1)} \cdots u_{n_1}^{(1)} \\ u_1^{(2)} \cdots u_{n_2}^{(2)} \\ \cdots \\ \cdots \\ u_1^{(m)} \cdots u_{n_m}^{(m)} \\ u_1^{(m+1)} \cdots u_{n_{m+1}}^{(m+1)} \end{array}$$

We then have:

$$\int \cdots N! \prod_I \prod_i \quad \mathrm{e}^{2\pi(N-(N_{m+2-I}+i)+K_I)a_i^{(I)}} \times \\ \times \left(\prod_I \prod_{i<j} (\mathrm{e}^{2\pi a_i^{(I)}} - \mathrm{e}^{2\pi a_j^{(I)}}) \prod_{I<J} \prod_{i,j} (\mathrm{e}^{2\pi a_i^{(I)}} - \mathrm{e}^{2\pi a_j^{(J)}}) \right)^{-1} \tag{2.54}$$

Now we can use the identity

$$\prod_{I<J} \prod_{i,j} (\mathrm{e}^{2\pi a_i^{(I)}} - \mathrm{e}^{2\pi a_j^{(J)}}) = \mathrm{e}^{2\pi \sum_I (N-N_{m+1-I}) \sum_i a_i^{(I)}} \prod_{I<J} \prod_{i,j} (1 - \mathrm{e}^{2\pi(a_j^{(J)} - a_i^{(I)})}) \tag{2.55}$$

to obtain

$$\int \cdots N! \prod_I \prod_i \mathrm{e}^{2\pi(n_I - i + K_I)a_i^{(I)}} \prod_{I<J} \prod_{i,j} (1 - \mathrm{e}^{2\pi(a_j^{(J)} - a_i^{(I)})})^{-1} \prod_I \prod_{i<j} (\mathrm{e}^{2\pi a_i^{(I)}} - \mathrm{e}^{2\pi a_j^{(I)}})^{-1} \tag{2.56}$$

We can again manipulate the dummy 'a' variables to write

$$\int [\mathrm{d}a] \cdots N! \prod_I \prod_i \mathrm{e}^{K_I 2\pi a_i^{(I)}} \sum_{\sigma \in S_N} \frac{\prod_i \mathrm{e}^{2\pi(n_I - i)a^I_{\sigma(i)}}}{\prod_{i<j}(\mathrm{e}^{2\pi a^I_{\sigma(i)}} - \mathrm{e}^{2\pi a^I_{\sigma(j)}})} \prod_{I<J} \prod_{i,j} (1 - \mathrm{e}^{2\pi a_j^{(J)} - 2\pi a_i^{(I)}})^{-1} \tag{2.57}$$

But using the identity for the determinant we stated earlier, the sum in the second term here is equal to one. Including the Vandermonde determinant, we can write our final result for the matrix model partition function as

$$\int \left(\prod_{I,i} \mathrm{d}a_i^{(I)}\right) \cdots N! \prod_I \left(\prod_i \mathrm{e}^{K_I 2\pi a_i^{(I)}} \prod_{i<j} (a_i^{(I)} - a_j^{(I)})^2\right) \times \tag{2.58}$$
$$\times \prod_{I<J} \left(\frac{\prod_{i,j}(a_i^{(I)} - a_j^{(J)})^2}{\prod_{i,j}(1 - \mathrm{e}^{2\pi a_j^{(J)} - 2\pi a_i^{(I)}})}\right) .$$

The new terms in this expression introduced by the trace insertion are those involving exponentials: there is an extra force on each block of eigenvalues (labelled by I) proportional to K_I, and there is an inter-block interaction.

So we write down the matrix model to be solved to evaluate large tableaux in the $\mathcal{N} = 2$ theory:

$$\langle W_R \rangle = \frac{1}{\mathcal{Z}} \int \prod_I \left(\prod_i \mathrm{d}a_i^{(I)} \mathrm{e}^{-\frac{8\pi^2 N}{\lambda}(a_i^{(I)})^2 + 2\pi K_I a_i^{(I)}} H^{2N}(a_i^{(I)}) \prod_{i<j} \frac{(a_i^{(I)} - a_j^{(I)})^2}{H^2(a_i^{(I)} - a_j^{(I)})}\right)$$
$$\times \prod_{I<J} \prod_{i,j} \left(\frac{(a_i^{(I)} - a_j^{(J)})^2}{\left(1 - \mathrm{e}^{2\pi(a_j^{(J)} - a_i^{(I)})}\right) H^2(a_i^{(I)} - a_j^{(J)})}\right) . \tag{2.59}$$

This has saddle point equations:

$$\frac{8\pi^2}{\lambda} a_i^{(I)} - 2\pi \frac{K_I}{2N} - K(a_i^{(I)}) = \frac{1}{N} \sum_{i(\neq j)} \left(\frac{1}{a_i^{(I)} - a_j^{(I)}} - K(a_i^{(I)} - a_j^{(I)})\right) \tag{2.60}$$
$$+ \frac{1}{N} \sum_{J(\neq I)} \sum_j \left(\frac{1}{a_i^{(I)} - a_j^{(J)}} - K(a_i^{(I)} - a_j^{(J)}) - \frac{1}{2} \frac{1}{1 - \mathrm{e}^{2\pi a_i^{(I)} - 2\pi a_j^{(J)}}}\right) .$$

If the model behaves qualitatively like that for $\mathcal{N} = 4$ SYM, the different groups of eigenvalues will be situated at intervals along the real line, with the $a_i^{(1)}$ furthest to the right, and each group having a profile as for the model without the trace insertion. This picture should be correct if one could establish that the eigenvalue spread within each group ($\sim \log \lambda$) were parametrically smaller than the spacings between groups. This would be an interesting avenue for future work. For $\mathcal{N} = 4$ this model yields a prediction for the IIB supergravity action evaluated on a weakly curved 'bubbling geometry'. The equivalent here would be informative about the curvatures involved in a gravity dual.

2.8 $N_f < 2N$ SQCD

Now we consider more general numbers of flavours, which we take to obey $N_f < 2N$ in order to preserve asymptotic freedom. Note that these theories are no longer scale-invariant, and the 't Hooft coupling needs to be renormalized as in section 1.6. Thus the coupling is no longer a free parameter - $\lambda \gg 1$ corresponds to the IR physics $R \gg \Lambda$, while small λ is the UV.

2.8.1 Pure $\mathcal{N} = 2$ SYM

The large-N saddle point of Pestun's model for pure $\mathcal{N} = 2$ SYM has the scaling form

$$\rho(x) = \frac{1}{\mu}\tilde{\rho}(y) \qquad x \equiv \mu y$$
$$\tilde{\rho}(y) = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}$$

where the end point μ of the distribution is related to the S^4 radius R and the dynamical scale Λ by $\mu = 2\mathrm{e}^{-1-\gamma}R\Lambda \equiv c_0 R\Lambda$. The (anti-)symmetric loop is computed by

$$\mathrm{Tr}_{S_k,A_k}[U] = \frac{1}{2\pi i}\oint \mathrm{d}t\ \exp \mp N\left(\int_{-\mu}^{\mu} \rho(x)\log(1 \mp \mathrm{e}^{2\pi x}t) - f\log t\right)$$

Let us consider the antisymmetric representation. The saddle point condition becomes

$$\int \mathrm{d}y \frac{\tilde{\rho}(y)}{\mathrm{e}^{-2\pi\mu y}t^{-1} + 1} = f$$

Now let $t = \mathrm{e}^{-2\pi\mu z}$:

$$\int \mathrm{d}y \frac{\tilde{\rho}(y)}{\mathrm{e}^{2\pi\mu(z-y)} + 1} = f \qquad .$$

Let us study the IR dynamics, so that $\mu \gg 1$. In this limit the saddle condition becomes

$$\frac{1}{\pi}\int_z^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}} = f \tag{2.61}$$

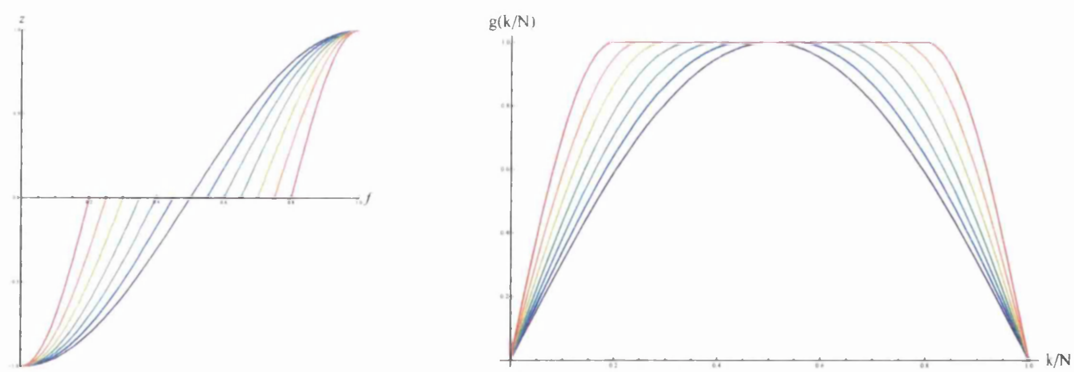
which is solved by $z = \sin\pi(\frac{1}{2} - f)$.

Now, according to the saddle point approximation, we evaluate the exponent on the saddle point. At large μ this is

$$\begin{aligned}
\langle \mathrm{Tr}_{S_k,A_k}[U] \rangle &= \exp 2\pi\mu N \left(\int_z^1 \mathrm{d}y \tilde{\rho}(y)\, y \right) \\
&= \exp\left(2\pi\mu N \sqrt{1 - z^2} \right) \\
&= \exp 2\pi c_0 N R \Lambda \sin(\pi \frac{k}{N})
\end{aligned}$$

Thus the pure $\mathcal{N} = 2$ antisymmetric loop ('k-string') exhibits the well-known sine law behaviour. Note also that it obeys an perimeter law rather than an area law. The same dependence on k/N was found for the k-string tension by Douglas and Shenker. There, the moduli space froze into this configuration by the introduction of an $\mathcal{N} = 1$ breaking mass term. Here the moduli space is lifted by the compact space (conformal mass of the adjoint scalars), but the same distribution results.

Figure 2.3: Saddle point and VEV for the SQCD Wilson loop



(a) The saddle point as a function of $f \equiv k/N$, for a range of α from $\alpha = 0$ (violet) through to $\alpha = 0.6$ (red)

(b) The exponent $g(k/N)$ in the antisymmetric Wilson loop expectation value, for the same range of α

2.8.2 Massless flavours

Now let us add N_f massless hypermultiplets in the fundamental representation. It was shown in [56] that the large-λ saddle point eigenvalue distribution is changed from the Yang-Mills case by the clumping of N_f of the eigenvalues in a delta function at the origin. It is interesting to note that this is exactly the point on the moduli space where classically the hypermultiplets are massless and the Coulomb branch joins on to a Higgs branch. Explicitly, we have ($\alpha \equiv \frac{N_f}{2N_c}$)

$$\tilde{\rho}(y) = (1-\alpha)\frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} + \alpha\,\delta(y)$$

so that the IR saddle equation (2.61) becomes

$$(1-\alpha)\frac{1}{\pi}\int_z^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}} + \alpha\,\Theta(z) \;=\; f$$

where $\Theta(z)$ is the Heaviside step function. The solution is plotted in fig. 2.3a. The VEV it gives rise to is given by

$$\langle \mathrm{Tr}_{S_k,A_k}[U]\rangle \;=\; \exp 2\pi c_0 N R\Lambda\, g(f)$$

where

$$g(f) \equiv \begin{cases} \sin(\pi\frac{f}{1-\alpha}) & f < \frac{1-\alpha}{2} \\ 1 & \frac{1-\alpha}{2} < f < \frac{1+\alpha}{2} \\ \sin(\pi\frac{f-\alpha}{1-\alpha}) & f > \frac{1+\alpha}{2} \end{cases}$$

is plotted in fig. 2.3b.

2.9 Conclusions

In this chapter we have studied large-rank Wilson loops in $\mathcal{N}=2$ SQCD, with massless flavours and for $N_f \leq 2N$. For the conformal case we found interesting behaviour which may shed light on a gravity dual - this was discussed in section 2.6. For pure SYM, we found a familiar sine law, while for general numbers of flavours we found interesting behaviour which it would be informative to study from the large-N limit of the Seiberg-Witten solution [57].

Chapter 3

Solving the BPS equations

The finite density physics of strongly interacting quantum systems presents significant challenges for existing theoretical frameworks. The questions of physical interest range from the behaviour of quark/baryonic matter at high densities [58–60] to the physics of quantum critical points in condensed matter systems [61]. In lattice approaches to QCD, a problem arises when considering a real chemical potential for the $U(1)_V$ symmetry under which opposite chiralities transform similarly $(q_{L,R} \mapsto e^{i\theta} q_{L,R})$, and whose charge is called 'baryon number' since mesons $(\bar{q}q)$ are invariant. This arises in the following way. QCD-like theories are generally quadratic in the fermions (in 4D, four-fermi interactions are irrelevant), and so it is natural first to perform the Gaussian integral so that one is left with an effective action only for the gauge fields. The fermion integral produces an insertion of the determinant of the quark kinetic operator

$$(i\Gamma^\mu \nabla_\mu - m - \mu \Gamma^0)$$

For real μ, this operator has complex eigenvalues, and so the standard lattice importance sampling approach does not work. One may attempt to cirumvent this in a number of ways [62], for example by the complex Langevin technique, which is an extension of a stochastic process incorporating complex actions. This has met with some success, however in order to study low temperature one needs increasingly large lattices which require considerable computing power [63]. There are interesting phenomena which are conjectured to occur in this region of the phase diagram [64], including dense and deconfined 'quark matter' phases about which little is known, and colour-flavour locked or colour superconducting phases - see figure 3.1.

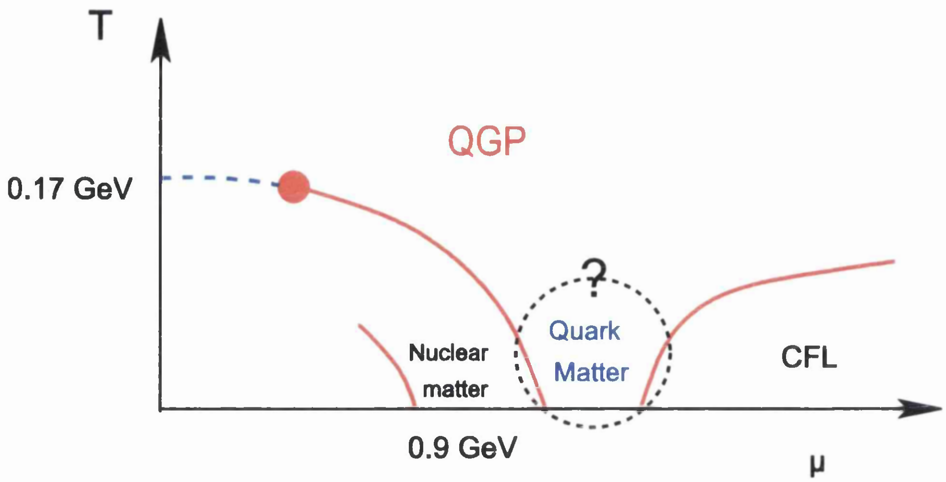


Figure 3.1: Cartoon phase diagram in the $\mu - T$ plane showing broadly expected features of QCD-like theories

AdS/CFT provides an avenue for tackling such phenomena within toy models that can be constructed and analyzed using the framework of gauge/string duality [65, 66]. One of the directions that has attracted considerable attention in recent years is the possible emergence of Lifshitz [67, 68] and Lifshitz-like (with hyperscaling violation) [5, 69, 70] scaling symmetries as infrared (IR) descriptions of strongly coupled field theories with holographic duals. A particularly fascinating feature of such scaling is the possibility of observing logarithmic violation of the area law for entanglement entropies indicating the presence of hidden Fermi surfaces [69, 70], and other possible signatures of holographic fermionic physics [71].

The aim of the next two chapters is to find supersymmetric backgrounds in string theory resulting from the backreaction of distributions of fundamental strings (F1-strings) within $AdS_5 \times S^5$ which as described in chapter 1 are dual to Wilson lines in $\mathcal{N} = 4$ SYM. These in turn can be interpreted as infinitely heavy quarks in the fundamental representation. We will find that this gives rise to (IR) geometries exhibiting a range of Lifshitz-like scalings with and without hyperscaling violation. Spatially uniform distributions of macroscopic strings correspond to a finite density of heavy quarks in the gauge theory[1]. It has been found in [72] that a specific non-supersymmetric

[1] It can also be argued that such configurations in the bulk should control the IR geometry produced by the backreaction of massive and massless flavour D7-branes with chemical potential for baryon

smearing of such string sources, preserving global $SO(6)$ symmetry, produces a flow from $\mathrm{AdS}_5 \times \mathrm{S}^5$ to a Lifshitz geometry with critical exponent $z = 7$ and a logarithmically running dilaton (see also [73]).

3.1 Finding supersymmetric solutions

The flow constructed in [72] has the perhaps conceptually disfavourable feature that the string sources are uniformly smeared so as to preserve $SO(6)$ symmetry. This is hard to understand from a field theoretic perspective; since it is not supersymmetric one might also ask whether the solution is stable. In particular, since the corresponding supersymmetric probe configuration is localized on the S^5 so as to break $SO(6) \rightarrow SO(5)$, it would be interesting to find what might be the supersymmetric geometries, and whether the $SO(6)$ solutions are unstable to flows towards them. In the following we tackle this problem by explicitly constructing such SUSY geometries. In this chapter we take a constructive approach: inspired by excellent work such as [23, 74], we write down the most general ansatz consistent with the symmetries of the problem, then solve the supersymmetry conditions within this ansatz. It is then left to solve those equations of motion which are not automatically satisfied.

3.2 BPS equations

Type IIB supergravity has 32 real supercharges parametrized by a complex chiral ten-dimensional spinor $\Gamma\epsilon = -\epsilon$. We begin by writing down the SUSY variations in the Einstein frame:

$$\delta_\epsilon \lambda = i(\Gamma \cdot P)\mathcal{B}^{-1}\epsilon^* - \frac{i}{24}(\Gamma \cdot G)\,\epsilon \tag{3.1}$$

$$\delta_\epsilon \psi_M = \mathrm{D}_M \epsilon + \frac{i}{480}(\Gamma \cdot F)\Gamma_M \epsilon - \frac{1}{96}\left[\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M\right]\mathcal{B}^{-1}\epsilon^* \quad . \tag{3.2}$$

We use the conventions of [23], which are stated explicitly in [75]. Note that the normalization of F_5 differs from that usually used in string theory by a factor of 4.

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number. We thank D. Mateos and J. Tarrio for stimulating discussions on this point.

Some notation

Type IIB supergravity is written in terms of two field strengths P and Q, and string theory in terms of the dilaton ϕ and axion $C^{(0)}$. Following e.g [74], we write the map between the supergravity and string theory variables:

$$P = \frac{1}{2}\mathrm{d}\phi + \frac{i}{2}\mathrm{e}^{\phi}\mathrm{d}C^{(0)} \tag{3.3}$$

$$Q = -\frac{1}{2}\mathrm{e}^{\phi}\mathrm{d}C^{(0)} \tag{3.4}$$

The supergravity equations of motion have a local $U(1)$ invariance with associated gauge field Q. Each field has a definite charge q under this $U(1)$: ϵ has $q = 1/2$, P has $q = 2$ and G_3 has $q = 1$. The field strengths have corresponding Bianchi identities written in terms of the $U(1)$-covariant derivative $\mathrm{D} \equiv \nabla - iqQ$

$$\mathrm{D}P = 0 \tag{3.5}$$

$$\mathrm{d}Q = -iP \wedge P^* \tag{3.6}$$

which are automatically satisfied when we use the map to string theory variables. This formulation comes from a gauge fixing of the version of the theory with an extra auxiliary scalar field, and the remnant of this is that each $SL(2,\mathbb{R})$ action is accompanied by a local $U(1)$ gauge transformation. This is the only way in which $SL(2,\mathbb{R})$ duality acts on the variables G, P, Q.

The ansatz we make is:

$$ds^2_{\text{Einstein}} = \mathrm{e}^{2A}dx^i dx^i + \mathrm{e}^{2B}d\Omega_4^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{3.7}$$

which describes a Euclidean 3-plane ($i, j, \cdots$ indices) and S^4 ($a, b, \cdots$ indices) both fibered over a base Lorentzian (2+1)D manifold which itself has a metric $g_{\mu\nu}$. This ansatz is invariant under the rotational and translational symmetries $ISO(3)$ of the $\mathbb{R}^3$, as well as the $SO(5)$ isometry group of the S^4.

For the other fields we choose the most general ansatz consistent with the same symmetries:

$$F_5 = \mathcal{F}_2 \wedge vol_{\mathbb{R}^3} + \mathrm{d}f \wedge vol_{S^4} \tag{3.8}$$

$$G_3 = g\, vol_{\mathbb{R}^3} + h\, vol_{\mathcal{M}_3} \tag{3.9}$$

$$P = P(x^\mu) \quad P_i = P_a = 0 \tag{3.10}$$

where $f \in \mathbb{R}, g, h \in \mathbb{C}$ are functions on $\mathcal{M}_3$, and $\mathcal{F}_2$ is a real two-form on $\mathcal{M}_3$. G_3 is a complex three-form which encodes both the RR and NS three-forms as (for example in the case of vanishing axion) $G_3 = \mathrm{e}^{-\phi/2} H_3 + i\,\mathrm{e}^{\phi/2} F_3$. Ten dimensional self-duality of F_5 implies that

$$\mathcal{F}_2 = \mathrm{e}^{(3A-4B)}(*_3\,\mathrm{d}f) \tag{3.11}$$

where $*_3$ is the Hodge star on $\mathcal{M}_3$.

The next step is to choose a basis of gamma matrices in ten dimensions. We choose:

$$\begin{array}{lcccccccc} \Gamma_i & = & \gamma_i & \otimes & \gamma_{S^4} & \otimes & \mathbb{1} & \otimes & \sigma^1 \\ \Gamma^a & = & \mathbb{1} & \otimes & \gamma^a & \otimes & \mathbb{1} & \otimes & \sigma^1 \\ \Gamma_\mu & = & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \gamma^\mu & \otimes & \sigma^2 \end{array}$$

where $\gamma_{S^4} \equiv +\gamma^6\gamma^7\gamma^8\gamma^9$ is the chirality matrix on S^4. The ten dimensional chirality matrix is $\Gamma = \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\sigma^3$, so that the IIB chirality condition reduces to

$$\sigma^3\epsilon = -\epsilon \tag{3.12}$$

We complete the basis by specifying gamma matrices within each factor space:

$$\begin{array}{lccccc} i: & \gamma^3 = \sigma^1, & \gamma^4 = \sigma^2, & \gamma^5 = \sigma^3 & & \\ a: & \gamma^6 = \sigma^1\otimes\mathbb{1}, & \gamma^7 = \sigma^2\otimes\mathbb{1}, & \gamma^8 = \sigma^3\otimes\sigma^1, & \gamma^9 = \sigma^3\otimes\sigma^2 & (\Rightarrow \gamma_{S^4} = -\sigma^3\otimes\sigma^3) \\ \mu: & \gamma^0 = i\sigma^2, & \gamma^1 = \sigma^1, & \gamma^2 = \sigma^3 & & \end{array}$$

Note that the basis for the gamma matrices on $\mathcal{M}_3$ is real (Majorana).

In this basis, the ten dimensional complex conjugation matrix $\mathcal{B}$, defined by $\{\mathcal{B}\Gamma^\mu\mathcal{B}^{-1} = (\Gamma^\mu)^*,\ \mathcal{B}^*\mathcal{B} = \mathbb{1}\}$, is

$$\mathcal{B} = \sigma^2\otimes(\sigma^2\otimes\sigma^1)\otimes\mathbb{1}\otimes\sigma^3 \tag{3.13}$$

$$= b_3\otimes b_4\otimes\mathbb{1}\otimes\sigma^3 \tag{3.14}$$

where b_3 and b_4 are charge conjugation matrices in $\mathbb{R}^3$ and S^4 respectively:

$$b_3\gamma^i b_3^{-1} = -(\gamma^i)^* \qquad b_4\gamma^a b_4^{-1} = -(\gamma^a)^*$$

We plug our ansatz and our Clifford algebra basis into the IIB SUSY variations, and after a few pages of careful work we end up with the following set of BPS conditions on our ten dimensional complex spinor ϵ:

$$\not{P}\mathcal{B}^{-1}\epsilon^* - \frac{1}{4}(\mathrm{e}^{-3A}g\gamma_{S^4} - h)\epsilon = 0 \tag{3.15}$$

$$\frac{1}{3}\mathrm{e}^{-A}\gamma^i\tilde{\nabla}_i\epsilon + \frac{i}{2}\not{\partial}A\gamma_{S^4}\epsilon + \frac{1}{2}\mathrm{e}^{-4B}\not{\partial}f\epsilon - \frac{i}{16}(3\mathrm{e}^{-3A}g + h\gamma_{S^4})\mathcal{B}^{-1}\epsilon^* = 0 \tag{3.16}$$

$$\mathrm{e}^{-B}\tilde{\nabla}_a\epsilon - \frac{i}{2}\gamma_a\not{\partial}B\epsilon + \frac{1}{2}\mathrm{e}^{-4B}\gamma_a\not{\partial}f\gamma_{S^4}\epsilon - \frac{i}{16}\gamma_a(\mathrm{e}^{-3A}g\gamma_{S^4} - h)\mathcal{B}^{-1}\epsilon^* = 0 \tag{3.17}$$

$$\mathrm{D}_\mu\epsilon + \frac{i}{2}\mathrm{e}^{-4B}\not{\partial}f\gamma_{S^4}\gamma_\mu\epsilon + \frac{1}{16}(\mathrm{e}^{-3A}g\gamma_{S^4} + 3h)\gamma_\mu\mathcal{B}^{-1}\epsilon^* = 0 \tag{3.18}$$

where here and in the following D_μ and ∇_μ denote derivatives on $\mathcal{M}_3$, and $\tilde{\nabla}_{i,a}$ are derivatives on $\mathbb{R}^3$ and S^4 respectively. Also note that when an operator appears which naturally acts within only one Clifford subspace, it should be taken as the tensor product with the identity matrix in the other tensor factors. For example, by $\gamma_{S^4}\epsilon$ we mean $(\mathbb{1}\otimes\gamma_{S^4}\otimes\mathbb{1}\otimes\mathbb{1})\epsilon$.

To proceed we must make an ansatz for the form of the ten dimensional spinor ϵ

$$\epsilon = \eta^\alpha \otimes \chi_a^\beta \otimes \epsilon_a^{\alpha\beta} \otimes \theta_a^{\alpha\beta} . \tag{3.19}$$

All repeated indices are to be summed over. The η_α are the linearly independent constant spinors on $\mathbb{R}^3$, $\alpha = 1, 2$, and the χ^β are the two sets of linearly independent Killing spinors on S^4, $\beta = 1, 2, 3, 4$, which can be taken to satisfy

$$\tilde{\nabla}_b \chi_a^\beta = \frac{a}{2}\gamma_{S^4}\gamma_b \chi_a^\beta \qquad \gamma_{S^4}\chi_a^\beta = \chi_{-a}^\beta \tag{3.20}$$

where we hope it is clear that the $a = \pm$ appearing here is not a spacetime index on S^4 but rather a label of the two different signs in the Killing spinor equation. The $\epsilon_a^{\alpha\beta}$ are commuting spinors on $\mathcal{M}_3$, and the $\theta_a^{\alpha\beta}$ are two-component spinors. The chirality condition (3.12) implies that $\sigma^3\theta_a^{\alpha\beta} = -\theta_a^{\alpha\beta}$, so that without loss of generality we can set

$$\theta_a^{\alpha\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \forall\, a, \alpha, \beta .$$

Following [23], we also note that, again without loss of generality, we can impose a reality condition on the basis Killing spinors[1]. Specifically, we impose[2]

$$(b_3 \otimes b_4)(\eta^* \otimes \chi_a^*) \quad = \quad \eta \otimes \chi_{-a} \tag{3.21}$$

[1] We thank John Estes for pointing this out.

[2] It is impossible to impose $b_3\eta^* = \eta$ on our basis, since $b_3^* b_3 = -1$ (there are no Majorana spinors in three Euclidean dimensions.) Furthermore we see that we cannot impose $\gamma_{S^4}\chi_a = \chi_{-a}$ and $b_4\chi_a = \chi_{-a}$ simultaneously, since $(\gamma_{S^4} b_4)^*(\gamma_{S^4} b_4) = -1$. However, we can impose a reality condition on the whole basis (rather than each factor indivually), since $(b_3\gamma_{S^4} b_4)^*(b_3\gamma_{S^4} b_4) = 1$.

We can now reduce the BPS to three dimensions, by writing them in terms of the two complex two-component spinors $\epsilon_\pm$ on $\mathcal{M}_3$:

$$\begin{aligned}
2\not{P}\,\epsilon^*_{-a} - \frac{1}{2}\mathrm{e}^{-3A}g\,\epsilon_{-a} + \frac{1}{2}h\,\epsilon_a &= 0 \qquad (3.22)\\
\frac{i}{2}\not{\partial}A\epsilon_{-a} + \frac{1}{2}\mathrm{e}^{-4B}\not{\partial}f\epsilon_a - \frac{i}{16}(3\mathrm{e}^{-3A}g)\epsilon^*_{-a} - \frac{i}{16}h\epsilon^*_a &= 0 \qquad (3.23)\\
-\frac{a}{2}\mathrm{e}^{-B}\epsilon_{-a} + \frac{i}{2}\not{\partial}B\epsilon_a - \frac{1}{2}\mathrm{e}^{-4B}\not{\partial}f\epsilon_{-a} + \frac{i}{16}(\mathrm{e}^{-3A}g)\epsilon^*_a - \frac{i}{16}h\epsilon^*_{-a} &= 0 \qquad (3.24)\\
\mathrm{D}_\mu\epsilon_a + \frac{i}{2}\mathrm{e}^{-4B}\not{\partial}f\gamma_\mu\epsilon_{-a} + \frac{1}{16}(\mathrm{e}^{-3A}g)\gamma_\mu\epsilon^*_a + \frac{1}{16}3h\gamma_\mu\epsilon^*_{-a} &= 0 \qquad (3.25)
\end{aligned}$$

Since no operators which affect the $\alpha, \beta, \cdots$ indices appear in (3.18), these indices can be omitted, with the understanding that there is a $2 \times 4 = 8$-fold multiplicity in each set of solutions $\{\epsilon_+, \epsilon_-\}$ we shall find of (3.22)-(3.25).

It is convenient to introduce a 'tau-matrix' notation for these equations, as follows:

$$(\tau^I\epsilon)_a \equiv \tau^I_{ab}\epsilon_b \quad I = 0,1,2,3 \qquad (3.26)$$

where $\tau^{1,2,3}$ are the usual Pauli matrices acting on the a, b indices, and $\tau^0 \equiv \mathbb{1}^{2\times 2}_{ab}$. Now the BPS equations reduced to $(2+1)$D read

$$\begin{aligned}
\not{P}\epsilon^* - \frac{1}{4}(\mathrm{e}^{-3A}g - h\tau^1)\epsilon = 0 &\qquad \text{(d)}\\
\not{\partial}A\epsilon - i\mathrm{e}^{-4B}\not{\partial}f\tau^1\epsilon - \frac{1}{8}(3\mathrm{e}^{-3A}g + h\tau^1)\epsilon^* = 0 &\qquad \text{(i)}\\
-\mathrm{e}^{-B}\tau^2\epsilon + \not{\partial}B\epsilon + i\mathrm{e}^{-4B}\not{\partial}f\tau^1\epsilon + \frac{1}{8}(\mathrm{e}^{-3A}g - h\tau^1)\epsilon^* = 0 &\qquad \text{(a)}\\
\nabla_\mu\epsilon - \frac{i}{2}Q_\mu\epsilon + \frac{i}{2}\mathrm{e}^{-4B}\not{\partial}f\gamma_\mu\tau^1\epsilon + \frac{1}{16}(\mathrm{e}^{-3A}g + 3h\tau^1)\gamma_\mu\epsilon^* = 0 &\qquad (\mu)
\end{aligned}$$

3.3 Spinor bilinear analysis

In this appendix we solve the BPS system (d)-(μ) using the standard techniques of bilinear analysis.

3.3.1 Bilinears

We introduce the real bilinears

$$f^{(I)} \equiv \epsilon^\dagger\sigma^2\tau^I\epsilon \qquad V^{(I)}_\mu \equiv i\,\epsilon^\dagger\sigma^2\tau^I\gamma_\mu\epsilon \quad . \qquad (3.27)$$

Likewise we have the complex bilinears

$$\tilde{f}^{(I)} \equiv \epsilon^t \sigma^2 \tau^I \epsilon \qquad \tilde{V}_\mu^{(I)} \equiv \epsilon^t \sigma^2 \tau^I \gamma_\mu \epsilon \quad . \tag{3.28}$$

Note that $\tilde{V}_{(2)}$ and $\tilde{f}^{(0,1,3)}$ all vanish identically since they are of the form $\epsilon^t M \epsilon$ where M is an antisymmetric matrix, and we have taken ϵ to be commuting. For typographical clarity, we use the (I) symbols as both subscripts and superscripts, but we intend no difference in meaning. The real bilinears have $q = 0$, and the complex ones have $q = 1$. We split the analysis into two, as is typical:

- On the one hand we have algebraic equations among the bilinears implied by the BPS equations. We will use these to define a preferred orthonormal basis for the tangent space of $\mathcal{M}_3$, namely an identity structure, and we express the fluxes in terms of this.
- On the other hand there are differential equations which give the 'torsion' of the identity structure, and which we use to define local coordinates and a metric.

At various points we will use the 3D Fierz identities (see section A.7), which express linear dependence between the bilinears. These are collected in full in appendix C.

3.3.2 Algebraic constraints

The first step is to the reduce the BPS equations to conditions on the minimum number of bilinears. We look at $\epsilon^\dagger \sigma^2$[(i)+(a)] and take real and imaginary parts to find that $f^{(2)} = 0$ and $V^{(0)} \cdot \partial B = 0$. Now taking $\epsilon^\dagger \tau^{0,1} \sigma^2$ {(i),(d)*}, we find

$$V^{(0)} \cdot X = 0 \qquad X = \mathrm{d}A,\, \mathrm{d}B,\, \mathrm{d}f,\, P \tag{3.29}$$

Next $\epsilon^\dagger \sigma^2 \tau^2 \gamma_\mu$[(i)-(a)]+c.c. gives

$$-4\,\mathrm{e}^{-4B} \mathrm{d}f\, f^{(3)} = 0$$

which is only solved for $f^{(3)} = 0$, assuming $\mathrm{d}f \neq 0$. Since $\mathrm{d}f = 0$ would lead to solutions preserving more supersymmetry than the 8 SUSYs we are interested in (for example the D1-brane solution), we ignore this possibility. Finally, using Fierz identities we can now show that

$$f^{(0)} = 0 \qquad V^{(1)} = 0 \tag{3.30}$$

$$V_{(2,3)}^2 = -V_{(0)}^2 = f_{(1)}^2 \qquad V^{(I)} \cdot V^{(J)} = 0 \quad \forall\, I \neq J \quad . \tag{3.31}$$

Now we have simplified things considerably. In particular we see that $\{\frac{1}{f^{(1)}}V_{(0)}, \frac{1}{f^{(1)}}V_{(3)}, \frac{1}{f^{(1)}}V_{(2)}\} \equiv \{e^0, e^1, e^2\}$ form an orthonormal basis for the cotangent space (and by raising indices, for the tangent space), and so we can take them as the vielbeine on $\mathcal{M}_3$.

Thus the mininum set of bilinears to consider consists of one real scalar, one complex scalar, and three real vectors: $\left\{f_{(1)}, \tilde{f}_{(2)}, V^{(0)}, V^{(2)}, V^{(3)}\right\}$. We next find expressions for the fluxes in terms of them.

3.3.2.1 Complex 3-form

Since $\tilde{f}^{(2)}$ is the only complex bilinear, we expect its phase to control the phases of h, g and P. Taking $\epsilon^\dagger\tau^{3,2}\sigma^2(\mathrm{d})^*$ and solving for g and h, we find that

$$h = \frac{4}{\tilde{f}^{(2)}}V^{(3)} \cdot P \tag{3.32}$$

$$\mathrm{e}^{-3A}g = \frac{4i}{\tilde{f}^{(2)}}V^{(2)} \cdot P \quad . \tag{3.33}$$

3.3.2.2 Five-form

This can be obtained by taking $\epsilon^\dagger\sigma^2\gamma_\mu(\mathrm{i})+(\mathrm{i})^\dagger\sigma^2\gamma_\mu\epsilon$:

$$-2i\, f^{(1)}\,\mathrm{e}^{-4B}\partial_\mu f + 2\partial_\nu A\epsilon^\dagger\sigma^2\gamma_\mu{}^\nu\epsilon + \frac{1}{8}\left((3\mathrm{e}^{-3A}g\tilde{V}_\mu^{(0)*} + h\tilde{V}_\mu^{(1)*}) - \mathrm{h.c}\right) = 0 \tag{3.34}$$

Completeness of the tangent space implies that the $\tilde{V}^{(I)}$ are linear combinations of the other vector bilinears, and indeed a Fierzing gives $\tilde{V}^{(0)} = \frac{\tilde{f}^{(2)}}{f^{(1)}}V^{(3)}$, $\tilde{V}^{(1)} = i\frac{\tilde{f}^{(2)}}{f^{(1)}}V^{(2)}$. This leads to

$$f_{(1)}^2\,\mathrm{e}^{-4B}\mathrm{d}f = \left[V^{(2)}\cdot\left(\mathrm{d}A + \frac{3}{2}\mathrm{e}^{-2i\theta}P\right)\right]\,V^{(3)} - \left[V^{(3)}\cdot\left(\mathrm{d}A + \frac{1}{2}\mathrm{e}^{-2i\theta}P\right)\right]\,V^{(2)} \tag{3.35}$$

where we have defined $\mathrm{e}^{i\theta}$ to be the phase of $\tilde{f}_{(2)}$.

3.3.2.3 Axiodilaton

We take the three combinations $\epsilon^\dagger\sigma^2\tau^{2,3,0}((\mathrm{i})+(\mathrm{a}))$. Plugging in the expressions for the fluxes obtained above we get

$$\begin{aligned} V^{(0)}\cdot\left[\partial(A+B) + \mathrm{e}^{-2i\theta}P\right] &= 0 \\ V^{(2)}\cdot\left[\partial(A+B) + \mathrm{e}^{-2i\theta}P\right] &= 0 \\ V^{(3)}\cdot\left[\partial(A+B) + \mathrm{e}^{-2i\theta}P\right] &= \mathrm{e}^{-B}f^{(1)} \end{aligned} \tag{3.36}$$

Using the orthonormality of our tangent space basis, the above equations imply that

$$\mathrm{e}^{-2i\theta}P = \frac{\mathrm{e}^{-B}}{f^{(1)}}V^{(3)} - \mathrm{d}(A+B) \quad . \tag{3.37}$$

We can therefore see that $\mathrm{e}^{-2i\theta}P$ is real (this is discussed in section 3.3.4). This implies that $P = \mathrm{e}^{2i\theta}\tilde{P}$, where $\tilde{P}$ is a real one-form.

As a last piece of information to take from the algebraic conditions, we take $\epsilon^t\sigma^2[(i)+(a)]$ and use the expressions for the fluxes to find that $|\tilde{f}_{(2)}|^2 = f_{(1)}^2$, so we can write

$$\tilde{f}^{(2)} = \mathrm{e}^{i\theta} f^{(1)} \quad . \tag{3.38}$$

In summary, we have defined an identity structure, found the fluxes (3.32),(3.33),(3.35),(3.37) in terms of it, and obtained the relation (3.38).

3.3.3 Torsion

In section 3.3.2 we reduced the problem of solving the BPS equations to finding two scalars $\{f_{(1)}, \tilde{f}_{(2)}\}$ and three vector bilinears $\{V^{(0)}, V^{(2)}, V^{(3)}\}$. These satisfy a system of differential equations which is implied by the BPS equations:

$$\mathrm{d}(\mathrm{e}^{A+2B}f^{(1)}) = 2\,\mathrm{e}^{A+B}V^{(3)} \tag{3.39}$$

$$\mathrm{D}(\mathrm{e}^{A+2B}\tilde{f}^{(2)}) = 2\,\mathrm{e}^{A+B}\tilde{V}^{(0)} \tag{3.40}$$

$$\mathrm{d}(\mathrm{e}^{2A+4B}V^{(0)}) = -4\,\mathrm{e}^{2A+3B} * V^{(2)} \tag{3.41}$$

$$\mathrm{d}(\mathrm{e}^{2A+4B}V^{(2)}) = -2\,\mathrm{e}^{2A}\mathrm{d}f \wedge V^{(3)} - 4\,\mathrm{e}^{2A+3B} * V^{(0)} \tag{3.42}$$

$$\mathrm{d}(\mathrm{e}^{2A+4B}V^{(3)}) = 2\,\mathrm{e}^{-4B}\mathrm{d}f \wedge V^{(2)} \tag{3.43}$$

We begin by showing that $V^{(0)}$ is a timelike Killing vector. It is timelike since $V_{(0)}^2 = -f_{(1)}^2$ is negative, and it satisfies

$$\nabla_{(\mu}V^{(0)}_{\nu)} = -g_{\mu\nu}V^{(0)} \cdot \partial(A+2B) = 0 \quad , \tag{3.44}$$

where the last equality follows from (3.29). Therefore $V^{(0)}$ satisfies the Killing equation on $\mathcal{M}_3$, and together with (3.29) this implies it is a Killing vector of the whole 10D metric. We define the coordinate t such that $\partial/\partial t = V_{(0)}^{\#}$, where the notation denotes $V^{(0)}$ as a vector.

We would like to find out about the phase $\mathrm{e}^{i\theta}$, which as we have already seen governs the phases of the complex fields (g, h, P). First we Fierz $\tilde{V}^{(0)}$, and then equate the LHSs of (3.40) and (3.39). Together with (3.38) this gives $Q = \mathrm{d}\theta$, so that the phase is just the $U(1)$ holonomy.

Next we consider the $\mathrm{d}f^{(1)}$ equation (3.39). This implies that $\mathrm{e}^{A+B}V^{(3)}$ is a closed form, and defining a local coordinate $y^2 \equiv \mathrm{e}^{A+2B}f^{(1)}$ we have that

$$V^{(3)} = y\,\mathrm{e}^{-(A+B)}\mathrm{d}y \quad . \tag{3.45}$$

Using (3.35) and (3.36), the $\mathrm{d}V^{(2)}$ equation (3.42) becomes

$$\mathrm{d}V^{(2)} = 3\left(\frac{\mathrm{d}y}{y} - \partial_y(A+B)\mathrm{d}y\right) \wedge V^{(2)} \tag{3.46}$$

so that we can write

$$V^{(2)} = y^3\mathrm{e}^{-3(A+B)}\mathrm{d}x \qquad . \tag{3.47}$$

We take x to be the final coordinate. We have automatically $\partial/\partial x \cdot \partial/\partial y = 0$.

Lastly we turn to the equation for $\mathrm{d}V^{(0)}$, which reads

$$\mathrm{d}V^{(0)} = (-2\,\mathrm{d}(A+2B) + 4\frac{\mathrm{d}y}{y}) \wedge V^{(0)} \tag{3.48}$$

so that

$$V^{(0)} = y^4\mathrm{e}^{-2A-4B}(\mathrm{d}t + \omega) \tag{3.49}$$

for some closed form $\mathrm{d}\omega = 0$.

In summary, we can now write down the full form of the metric in terms of the warp factors $\mathrm{e}^A, \mathrm{e}^B$:

$$\boxed{\mathrm{d}s_3^2 = -y^4\mathrm{e}^{-2A-4B}(\mathrm{d}t+\omega)^2 + \frac{\mathrm{e}^{2B}}{y^2}\mathrm{d}y^2 + y^2\,\mathrm{e}^{-4A-2B}\mathrm{d}x^2} \tag{3.50}$$

and the fluxes are

$$P = \mathrm{e}^{2i\theta}\mathrm{d}(\log y - A - B) \tag{3.51a}$$

$$g = -4i\,\mathrm{e}^{i\theta}\mathrm{e}^{5A+B}y^{-1}\partial_x(A+B) \tag{3.51b}$$

$$h = 4\,\mathrm{e}^{i\theta}\mathrm{e}^{-B}y\,\partial_y(\log y - A - B) \tag{3.51c}$$

$$\mathrm{d}f = \frac{y^4}{4}\left[\partial_x(y^{-6}\mathrm{e}^{2A+6B})\,\mathrm{d}y + \partial_y(y^{-2}\mathrm{e}^{-2A+2B})\,\mathrm{d}x\right] \tag{3.51d}$$

3.3.4 Reality condition and $SL(2,\mathbb{R})$

We now address the reality condition implied by (3.37):

$$\mathfrak{Im}(\mathrm{e}^{-2i\theta}P) = 0 \tag{3.52}$$

This implies that $P = \mathrm{e}^{2i\theta}\tilde{P}$, where $\tilde{P}$ is a real one-form. Then (3.5) and (3.6) imply $\mathrm{d}Q = 0$ and $Q = \mathrm{d}\theta$, so that Q is pure gauge (these relations are actually implied by the BPS equations, as shown in (3.3.3)). Therefore by a local $U(1)$ gauge transformation $\mathfrak{U} = \mathrm{e}^{-i\theta}$, we can map to real P, i.e. vanishing axion. We know this must be the accompanying gauge transformation to an $SL(2,\mathbb{R})$ action, so the BPS equations give just the solution with $C^{(0)} = 0$ and its orbit under S-duality, parametrized by θ. This situation is familiar from e.g. [23].

3.4 Equations of motion

Since we have found that our configurations have $C^{(0)} = 0$ up to a duality, we set $\theta = 0$ to zero in this section, with the knowledge that after dualizing we will also have solutions. We also set $\omega = 0$. Thus we have $P = \frac{1}{2}\mathrm{d}\phi$. We will use a slightly cleaner version of the BPS configuration which is valid in this case:

$$\mathrm{d}s^2 = -\mathrm{e}^{2(A+\phi)}\mathrm{d}t^2 + \mathrm{e}^{2A}\mathrm{d}x^i\mathrm{d}x^i + \mathrm{e}^{-2A}\left(\mathrm{e}^{-\phi}(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_4^2) + \mathrm{e}^{\phi}\mathrm{d}x^2\right) \tag{3.53}$$

$$F_5 = \frac{y^4}{4}(1+*)\left(-\partial_x(\mathrm{e}^{-4A-3\phi})\mathrm{d}y + \partial_y(\mathrm{e}^{-4A-\phi})\mathrm{d}x\right) \wedge vol(S^4) \tag{3.54}$$

$$H_3 = \partial_y(\mathrm{e}^{2\phi})\,\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}x \qquad F_3 = 2\,\mathrm{e}^{4A-\phi}\partial_x\phi\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \tag{3.55}$$

The purpose of this section is to show that once consistent string sources $\propto \rho(y)$ are added, all of the equations of motion and Bianchi identities are satisfied provided we have $F_0 = -2\mathrm{e}^{4A}\partial_x\mathrm{e}^{-\phi}$, where F_0 is a (non-zero) real constant, and the following equation, which we will refer to as the 'Poisson equation', is satisfied:

$$\frac{1}{y^4}\partial_y(y^4\partial_y\mathrm{e}^{-2\phi}) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi} = \rho(y) \quad . \tag{3.56}$$

The case $F_0 = 0$ will be dealt with in chapter 4.

3.4.1 F_3 Bianchi identity

This states that $F_3 = F_0 \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ for some constant F_0. Using the BPS expression for F_3, we obtain $F_0 = -2\mathrm{e}^{4A}\partial_x \mathrm{e}^{-\phi}$, which is the relationship between A and ϕ stated in the main text.

3.4.2 F_5 Bianchi identity

This reads

$$\mathrm{d}F_5 - \frac{1}{4}H_3 \wedge F_3 = 0 \tag{3.57}$$

Substituting in the solutions of the BPS equations we obtain explicit expressions for F_5

$$\mathrm{d}f = \frac{y^4}{4}\left[\partial_y(\mathrm{e}^{-4A-\phi})\mathrm{d}x - \partial_x(\mathrm{e}^{-4A-3\phi})\mathrm{d}y\right] \tag{3.58}$$

$$\Rightarrow \mathcal{F}_2 = \mathrm{e}^{3A-4B} *_3 \mathrm{d}f = -\mathrm{e}^{7A+2\phi}\frac{1}{4}\left(\mathrm{e}^{A}\partial_y(\mathrm{e}^{-4A-\phi})\mathrm{d}y + \mathrm{e}^{A+2\phi}\partial_x(\mathrm{e}^{-4A-3\phi})\mathrm{d}x\right) \wedge \mathrm{d}t \tag{3.59}$$

$$= \frac{1}{4}(\frac{1}{2}\mathrm{d}(\mathrm{e}^{4A+\phi}) + \mathrm{e}^{2\phi}2\mathrm{e}^{4A-\phi}\partial_x\phi\,\mathrm{d}x) \wedge \mathrm{d}t \tag{3.60}$$

The Bianchi identity comes in two pieces: a piece proportional to $vol(S^4)$, and one proportional to $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. We deal with the second part first. Using the F_3 Bianchi, we can express the two sides as follows:

$$\mathcal{F}_2 = \frac{1}{4}(\mathrm{d}(\mathrm{e}^{4A+\phi}) - \mathrm{e}^{2\phi}F_0\mathrm{d}x) \wedge \mathrm{d}t \tag{3.61}$$

$$\Rightarrow \mathrm{d}F_5 = \cdots + \frac{F_0}{4}\partial_y \mathrm{e}^{2\phi}\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{3.62}$$

$$\frac{i}{8}G \wedge G^* = \frac{1}{4}H_3 \wedge F_3 = \frac{1}{4}\partial_y \mathrm{e}^{2\phi}\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}x \wedge (F_0\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3) \tag{3.63}$$

so we see that this part is automatically satisfied. Now we turn to the first part. Setting this to zero amounts to saying we can locally find a function $f(x,y)$. Again using the F_3 Bianchi we have

$$\mathrm{d}f = -\frac{2}{F_0}\frac{y^4}{4}\left(\frac{1}{2}\partial_y\partial_x \mathrm{e}^{-2\phi}\mathrm{d}x - \frac{1}{4}\partial_x^2 \mathrm{e}^{-4\phi}\mathrm{d}y\right) \wedge vol(S^4) \quad . \tag{3.64}$$

Integrating this gives

$$\partial_x f = -\frac{1}{4}\frac{1}{F_0}\partial_x(y^4\partial_y \mathrm{e}^{-2\phi}) \tag{3.65}$$

$$f = -\frac{1}{4}\frac{1}{F_0}y^4\partial_y \mathrm{e}^{-2\phi} + g(y) \tag{3.66}$$

We now differentiate this last expression and set it equal to $\partial_y f$:

$$\partial_y f = -\frac{1}{4}\frac{1}{F_0}\partial_y(y^4\partial_y \mathrm{e}^{-2\phi}) + g'(y) \stackrel{!}{=} \frac{1}{8F_0}y^4\partial_x^2\mathrm{e}^{-4\phi} \tag{3.67}$$

$$\Rightarrow \frac{1}{y^4}\partial_y(y^4\partial_y\mathrm{e}^{-2\phi}) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi} = F_0\frac{4}{y^4}g'(y) \tag{3.68}$$

The last expression gives our Poisson equation when we identify $\rho(y) = F_0\frac{4}{y^4}g'(y)$.

3.4.3 H_3 equation of motion

For the first term, we can write

$$*H_3 = -\mathrm{e}^{4A-\phi}y^4\partial_y\mathrm{e}^{2\phi}\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge vol(S^4) \tag{3.69}$$

$$\Rightarrow \mathrm{d}(\mathrm{e}^{-\phi} * H_3) = \mathrm{d}(y^4\partial_y(\mathrm{e}^{-2\phi})\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge vol(S^4) \tag{3.70}$$

$$= \Big((\partial_y(y^4\partial_y(\mathrm{e}^{-2\phi})\mathrm{d}y + y^4\partial_x\partial_y(\mathrm{e}^{-2\phi})\mathrm{d}x\Big)\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge vol(S^4) \tag{3.71}$$

And for the the second term, using (3.58) we immediately have

$$4F_3\wedge F_5 = -2y^4\left(\frac{1}{2}\partial_x\partial_y\mathrm{e}^{-2\phi}\mathrm{d}x - \frac{1}{4}\partial_x^2\mathrm{e}^{-4\phi}\mathrm{d}y\right)\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge vol(S^4) \tag{3.72}$$

Finally we obtain

$$\mathrm{d}(\mathrm{e}^{-\phi} * H_3) + 4F_3\wedge F_5 = \left[\partial_y(y^4\partial_y\mathrm{e}^{-2\phi}) + \frac{y^4}{2}\partial_x^2\mathrm{e}^{-4\phi}\right]\mathrm{d}y\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge vol(S^4) \stackrel{!}{=} \Omega_8 \tag{3.73}$$

where is Ω_8 is an appropriate smeared source term.

3.4.4 Dilaton equation of motion

This is

$$\mathrm{d} * \mathrm{d}\phi + \frac{1}{2}G_3\wedge *G_3 = \frac{\mathrm{e}^{\phi}}{2y}\left(\frac{1}{y^4}\partial_y(y^4\partial_y\mathrm{e}^{-2\phi}) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\right) * 1 = 0 \quad . \tag{3.74}$$

3.4.5 Einstein equations

We quote the RHS side of the Einstein equations in the form

$$R_{MN} - \frac{1}{2}R\,g_{MN} + \text{flux terms} = T^{\text{sources}}_{MN}$$

and find that the sources must take the form

$$T_{ii}^{\text{sources}} = -\frac{e^{2\phi+6A}}{F_0 y}\partial_x(D[\phi]) \tag{3.75}$$

$$T_{aa}^{\text{sources}} = T_{yy}^{\text{sources}} = 0 \tag{3.76}$$

$$T_{00}^{\text{sources}} = -T_{xx}^{\text{sources}} = -\frac{e^{3\phi+2A}}{2}D[\phi] \tag{3.77}$$

where $D[\phi]$ refers to the LHS of the Poisson equation. Here we see that once we impose the Poisson equation, only T_{00}^{sources} and T_{xx}^{sources} are non-zero. It possible to show that a smeared distribution of strings stretched along the x axis with a smearing form $\propto \rho(y)$ gives precisely these source terms, as well as the correct source terms for the H_3 and dilaton equations of motion.

3.5 Conclusions

Motivated by finding solutions describing $SO(5)$ symmetric backreacted Wilson lines, in this chapter we found the most general solutions to the BPS equations of type IIB supergravity with $ISO(3) \times SO(5)$ symmetry. We then reduced the solution of the equations of motion to one condition.

Chapter 4

Lifshitz geometries and smeared strings

4.1 Introduction

Spatially homogeneous $(d+2)$-dimensional spacetime metrics exhibiting Lifshitz-like scaling with dynamical critical exponent z and hyperscaling violation have the form [5, 69, 70]

$$ds^2 = r^{-2\theta/d}\left(-r^{2z}\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2\mathrm{d}\vec{x}^2\right), \tag{4.1}$$

so that under the rescalings: $\vec{x} \mapsto a\,\vec{x}$, $r \mapsto a^{-1}r$ and $t \mapsto a^z\,t$, the line element transforms covariantly as $\mathrm{d}s \mapsto a^{\theta/d}\,\mathrm{d}s$.

Within the solutions of chapter 3, we will consider the case when $F_3 = 0$, and so encounter such scaling backgrounds with $d = 3$, preserving 8 supersymmetries, as IR descriptions of $\mathcal{N} = 4$ SYM at strong coupling with a smeared density of supersymmetric Wilson lines or heavy quarks. In the large-N limit, we take the number density of such heavy quarks to scale as N^2, so that it becomes necessary to include their backreaction on the field theory vacuum. The specific class of backgrounds that we consider in this chapter are those which have the form of certain well known intersecting brane configurations [76–78]. The configurations within this class can be thought of as pairs of quarks and antiquarks which are mutually supersymmetric (by having opposite internal $SO(6)$ orientations) and are distributed uniformly in the spatial directions of the gauge theory. A particular solution within this class has appeared in [79] where its Lifshitz-like scaling properties have been pointed out.

We find that one category of the intersecting F1-D3 configurations naturally leads to an exact scaling solution in the infrared, exhibiting Lifshitz-scaling with $z = 7$ which is mildly broken by a logarithmically running dilaton. The background preserves supersymmetry and an $SO(5)$ internal symmetry. This result is noteworthy since the $z = 7$ Lifshitz scaling behaviour matches that of [72], a $SO(6)$-preserving non-supersymmetric setup. This is indicative that the $z = 7$ scaling is probably a universal feature of $\mathcal{N} = 4$ SYM with finite quark density (at strong coupling and large-N). It points toward the possibility that when the quark flavours are made dynamical (by introducing D7-branes for instance [80]) and their backreaction taken into account at finite baryon density, then the low temperature IR physics may well be controlled by a similar scaling solution (see [81] for closely related discussions). A concrete framework where this possibility can be further investigated is the unquenched smeared flavour brane setup of [82–85].

Within the setting of the supersymmetric configurations in this paper, constructing the full flow from $\mathrm{AdS}_5 \times \mathrm{S}^5$ to the IR scaling solution remains a challenging task due to the reduced isometry of the dual gravity backgrounds. However, we analytically examine the departure of the flow induced by the supersymmetric smeared strings in the UV, and find close similarities to the $SO(6)$ symmetric case of [72] wherein it was possible to obtain the entire flow numerically.

The supersymmetric intersecting brane configurations present us with a further interesting route for obtaining non-trivial scaling solutions. This is because of the existence of supersymmetric moduli spaces at zero temperature. In particular, it is possible to move out to a generic point on the Coulomb branch moduli space (where the gauge group is generically Higgsed to a product of Abelian factors) of $\mathcal{N} = 4$ SYM and examine the effect of a finite quark density. In the dual gravity description this corresponds to a general distribution of N D3-branes at large-N. We consider such continuous distributions of the D3-branes and F1-strings that are compatible with the supersymmetries preserved by the intersecting brane configurations. Specifically, we introduce distributions for these sources with power law scalings in the IR and solve the supergravity equations for such power law density functions. We thus find a two parameter family of Lifshitz-like scaling solutions with hyperscaling violation coefficients. Imposing the weak energy condition on each of the source distributions restricts the allowed values of z and θ.

Perhaps the most interesting result within the class of Coulomb branch configurations is the appearance of a solution with $z, \theta \to \infty$ and $\eta \equiv -\theta/z$ fixed (satisfying $\eta \geq 1$). These give rise to geometries that are conformal to $\mathrm{AdS}_2 \times \mathbb{R}^3$ with vanishing entropy at zero temperature. Such geometries have been argued to be relevant for holographic descriptions of fermionic physics [71].

All the solutions we find bear close resemblance to scaling solutions in Einstein-Maxwell-dilaton theories [68, 86, 87]. In particular the dilaton softly breaks the scaling invariance via logarithmic running. Furthermore since the dilaton runs to zero in the IR, it renders α' corrections important in the deep IR.

4.2 Smeared strings and $\mathcal{N} = 4$ SYM

The Wilson lines we study in this chapter are the simplest cases within the ansatz (1.4), namely straight contours each preserving 16 of the 32 SUSYs and a subgroup $SO(5) \subset SO(6)_R$ of the R-symmetries. As shown in section 1.3.3, a configuration of such lines placed at different spatial points preserves 8 real supersymmetries.

Our goal in this chapter is to model a particular state with finite heavy quark density by taking a specific BPS Wilson line configuration and smearing it uniformly along all spatial directions of the $\mathcal{N} = 4$ theory. The unsmeared configuration consists of a mutually BPS quark-antiquark pair i.e. antiparallel lines placed at antipodal points of the internal S^5 [50, 88]. This preserves an $SO(5)$ global symmetry and, prior to taking the near horizon limit of the D3-branes, can be viewed as a supersymmetric F1-D3 intersection. A specific delocalized version of this intersection has been discussed by Dey and Roy [79] and shown to lead to Lifshitz-like scaling. The backreacted geometries resulting from the spatially smeared configurations are $\frac{1}{4}$-BPS, preserving 8 real supercharges.

We will consider two distinct classes of $\frac{1}{4}$-BPS configurations in this chapter:

- In the first category lies the so-called *partially localized* F1-D3 intersection [78], where F1-strings are smeared along the relative transverse directions (spatial coordinates) on the D3-branes, and the D3-branes are not smeared. We will argue that the corresponding gravity background represents a flow from $\mathrm{AdS}_5 \times \mathrm{S}^5$ to a $z = 7$ Lifshitz-like geometry.

- The second class involves delocalized D3-brane and F1-string distributions, allowing for a general smearing density for both sets of sources. The solution of [79] is a special case within this category of solutions. We find a large class of zero temperature scaling solutions with a range of dynamical critical exponents and hyperscaling violation coefficients.

4.2.1 Metric ansatz in 10D

Given the symmetries and supersymmetries of such configurations, we showed in chapter 3[1] that the type IIB fermionic variations vanish if one takes the following ansatz for the Einstein frame metric, the Ramond-Ramond fluxes F_3, F_5 and Neveu-Schwarz field strength H_3:

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Einstein}} &= -\mathrm{e}^{2(A+\phi)}\mathrm{d}t^2 + \mathrm{e}^{2A}\sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i + \mathrm{e}^{-2A+\phi}\mathrm{d}x^2 + \mathrm{e}^{-2A-\phi}\left(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_4^2\right) , \\
g_s F_5 &= \frac{y^4}{4}(1+*)\left[\partial_y\left(\mathrm{e}^{-4A-\phi}\right)\mathrm{d}x - \partial_x\left(\mathrm{e}^{-4A-3\phi}\right)\mathrm{d}y\right]\wedge vol(S^4) , \\
H_3 &= \partial_y\left(\mathrm{e}^{2\phi}\right)\,\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}x , \qquad F_3 = 2\,\mathrm{e}^{4A-\phi}\partial_x\phi\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 . \qquad (4.2)
\end{aligned}
$$

Here $\mathrm{d}\Omega_4^2$ is the metric of the unit radius four-sphere with volume form $vol(S^4)$; the warp factor A and dilaton ϕ are functions of the x and y coordinates only. The gauge theory lives in the 4D spacetime spanned by the coordinates (t, x^1, x^2, x^3). In addition, the ten-dimensional complex spinor ϵ parametrising supersymmetry variations must satisfy the projection conditions

$$
\Gamma^{\hat{t}\hat{x}}D^{-1}\epsilon^* = \epsilon , \qquad i\,\Gamma^{\hat{t}\hat{x}_1\hat{x}_2\hat{x}_3}\epsilon = \epsilon , \qquad (4.3)
$$

leading to $\frac{1}{4}$-BPS solutions. Here D is the ten dimensional complex conjugation matrix. The projections are those associated to fundamental strings and D3-branes, respectively.

Within this ansatz, the geometry can be viewed locally as a fibration of $\mathbb{R}_t\times\mathbb{R}^3\times S^4$ over the x-y plane. Therefore, these are time independent backgrounds with an $SO(5)$ isometry and symmetry under translations and rotations in $\mathbb{R}^3$.

[1] Also to appear in [3]

4.2.2 (Partially) Localized F1-D3 intersection

In this chapter we will only study solutions with $F_3 = 0$. This precludes a charge density for D5-branes that could be interpreted as baryon density as in [72]. It is also physically clear why this is the case: we are considering a uniform density of quark and *anti*-quark pairs (with opposite $SO(6)$ orientations). With this choice, the backgrounds represent F1-D3 BPS intersections of the kind analyzed in [76–78]. We can make contact with the standard form of the metric for such backgrounds with the identifications:

$$\mathrm{e}^{-2\phi} = h_1\,, \qquad \mathrm{e}^{-4A-\phi} = h_3\,, \tag{4.4}$$

where h_1 and h_3 are the harmonic functions associated to the F1-strings and D3-branes respectively. It is well known that in these types of solution (e.g. [78]) there is a smearing along the relative transverse directions of the strings/branes i.e. those which are parallel to the world volume of some of the branes (strings), but perpendicular to others. They are constructed by the so-called harmonic rule: to each type of brane (string) there is a corresponding function which is harmonic in the space transverse to their world-volumes. In order to source two sets of branes in the same solution one simply multiplies the two harmonic functions together in the appropriate way.

AdS$_5\times$S^5 vacuum: The vacuum solution, namely AdS$_5\times$S^5, is recovered when

$$h_1 = 1\,, \qquad h_3 = \mathrm{e}^{-4A} = \frac{1}{(x^2+y^2)^2}\,, \quad x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad 0\leq\theta\leq\pi\,. \tag{4.5}$$

This is indicated in figure 1. The five-sphere is obtained by fibering the S^4 along a semicircle enclosing the origin.

Backreaction of string sources: The presence of macroscopic string sources will lead generically to a non-vanishing NS-NS three-form flux H_3. The form of $H_3 = \mathrm{d}B_2$ in our ansatz (4.2) suggests that the strings must lie parallel to the x-axis (see figure 4.1). Now, both the functions h_3 and h_1 are non-trivial, and for sources localized in the x-y plane they are determined by the Bianchi identity for F_5 and equation of motion for H_3, namely,

$$\partial_y\left(y^4\partial_y h_3\right) + y^4\, h_1\partial_x^2 h_3 = 0\,, \qquad \partial_y(y^4\partial_y h_1) = 0\,. \tag{4.6}$$

Figure 4.1: <u>**Left:**</u> The x-y half-plane in AdS with $x = r\cos\theta$ and $y = r\sin\theta$ where r is the radial coordinate in AdS and θ the polar angle on the S^5. <u>**Right:**</u> D3-brane positions (in red) on the x-y plane, with infinite F1-strings oriented along the x-axis. The thickened lines indicate that the respective distributions may have a non-zero extent along the x-axis.

These are Laplace-like equations for (partially) localized sources smeared along certain directions transverse to them (the transverse S^4 and the spatial $\mathbb{R}^3$). Since $F_3 = 0$, we also know that h_1 is only a function of the y-coordinate,

$$e^{-2\phi} = h_1(y) = 1 + \frac{Q_1}{y^3}\,. \tag{4.7}$$

This is what we would expect for a localized, infinite string source oriented along the x-axis. The additive integration constant has been set to unity by the requirement that the dilaton ϕ vanishes for large y. Here Q_1 is proportional to the number density of F-strings n_{F1}, which also measures the density of heavy quark sources in the gauge theory:

$$Q_1 \equiv \sqrt{\lambda}\,\frac{n_{\mathrm{F1}}}{N^2}\,\frac{4\pi^2}{V_{S^4}}\,, \tag{4.8}$$

λ being the 't Hooft coupling of the $\mathcal{N} = 4$ theory and V_{S^4} the volume of S^4. The number density of heavy quarks n_{F1} must scale as N^2 in order to keep Q_1 fixed in the large-N limit, and to consistently include their backreaction on the background.

The normalizations are fixed by demanding that the equation of motion for H_3 yield a number density n_{F1} of macroscopic F-strings localized at $y = 0$,

$$\mathrm{d}(e^{-\phi} * H_3) = \frac{n_{\mathrm{F1}}}{2\pi\alpha'}\,16\pi G_N\,\delta(y)\,\mathrm{d}y \wedge \epsilon_{(4)} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,. \tag{4.9}$$

Setting the AdS-radius to unity, the 't Hooft coupling λ of the gauge theory is related to the tension of a fundamental string as $(2\pi\alpha')^{-1} = \frac{\sqrt{\lambda}}{2\pi}$. In addition, Newton's constant in ten dimensions is given by $16\pi G_N = (2\pi)^7 g_s^2\,\alpha'^4$, with $\lambda = 4\pi g_s N$.

With this solution for h_1, the full background is determined by $h_3(x, y)$, which solves ((4.6)) subject to the requirement that the geometry is asymptotically, locally

$AdS_5 \times S^5$. It is not possible to solve this equation analytically, but we can analyze its UV and IR asymptotics.

4.2.3 $z = 7$ Lifshitz IR

In the IR limit, which corresponds to small y, we have that $h_1 \approx Q_1/y^3$. The harmonic function $h_3(x, y)$ in this limit is [78]:

$$h_3(x,y) \;=\; \frac{1}{y^{5/2}}\left(x^2 y + 4Q_1\right)^{5/2} . \tag{4.10}$$

We may view this as a 'near horizon' limit for the F-strings. In fact h_3 can take the more general form, similar to that for multi-centred D3-brane sources,

$$h_3 \sim \sum_i P_i \left((x - x_i)^2 + 4Q_1/y\right)^{5/2}$$

but this generalization is not of particular interest to us at this point. It is easy to see that the solution for h_3 follows from a scaling argument[1]. The second important remark is that such solutions cannot correspond to localized F1-D3 intersections, since h_3 increases as a positive power of x for large x. The D3-branes are in fact delocalized along the x-axis in this limit $(y/{Q_1}^{1/3} \ll 1)$. The reasons behind such 'spontaneous delocalization' were explored in [89], including the case which is precisely the S-dual of the setup we are studying, namely D1-branes smeared along the spatial directions of D3-branes. In this case the spontaneous delocalization was attributed to the analogue of the Coleman-Mermin-Wagner theorem in the 0+1 dimensional quantum mechanics of the intersection.

One can confirm that the D3-branes are indeed not localized by noting that F_5 (in this limit) depends only on $v \equiv x^2 y$,

$$g_s F_5 \;=\; (1+*)\, \mathrm{d}f(v) \wedge \epsilon_{(4)}\,, \qquad\qquad v \equiv x^2 y\,, \tag{4.11}$$

$$\mathrm{d}f(v) \;\equiv\; -\frac{5Q_1}{4\sqrt{v}}\,(4Q_1 + v)^{5/2}\mathrm{d}v\,.$$

[1] Allowing for a source density $\rho(x, y)$ on the right hand side of (4.6), we expect that under $x \mapsto sx$, $y \mapsto s^a y$, for a scaling solution we should have $\rho \mapsto \rho/s^{1+a}$. Comparing the two sides we find that $a = -1/2$ and $h_3 = \tilde{h}(x^2y)/y^{5/2}$ for some function $\tilde{h}$. The differential equation for $\tilde{h}(v \equiv x^2y)$, yields the general solution $\tilde{h}(v) = c_1\,(v + 4Q_1)^{5/2} + c_2\,\sqrt{v}(v^2 + 10Qv + 30Q^2)$.

Therefore, the D3-brane charge can be computed by $\int_\gamma \mathrm{d}f(v)$ where γ is a path joining two points on the curves, say, $v = v_1$ and $v = v_2$. For any two points on the x-axis, which sits at $v = 0$, the integral along a path joining them vanishes trivially, and therefore the D3-brane charge is not localized at the origin, nor at any other finite point on the axis. It also follows, by allowing arbitary deformations of the curve, that there are no localized D3-brane sources for any finite values of x and y. In fact the sources must be viewed as being placed at infinity. This is also related to the fact that the S^4 in the geometry does not shrink anywhere in the x-y plane in this IR limit. This should be contrasted with $\mathrm{AdS}_5 \times \mathrm{S}^5$ wherein the D3-brane charge is obtained by choosing a path with end-points on the x-axis and, importantly, enclosing the origin, as in figure 4.1.

This picture therefore suggests that the F-strings have 'pulled apart' the D3-branes so that they appear delocalized for scales set by $y \ll {Q_1}^{1/3}$.

Plugging in the expressions for h_3 and h_1 in this IR limit, defining a new radial coordinate $\rho \equiv y^{1/4}$ and after appropriate coordinate rescalings, we find

$$\begin{aligned} \mathrm{d}s^2_{\text{Einstein}} &= Q_1^{3/2} \Big[\left(-\rho^{14}\,\mathrm{d}t^2 + \rho^2\,\mathrm{d}x^i\mathrm{d}x^i\right)(x^2\rho^4+4)^{-5/4} + 16\,\frac{\mathrm{d}\rho^2}{\rho^2}\,(x^2\rho^4+4)^{5/4} \\ &\quad + (\mathrm{d}\Omega_4^2 + \rho^4\mathrm{d}x^2)(x^2\rho^4+4)^{5/4} \,\Big] \,, \qquad (4.12) \\ \mathrm{e}^\phi &= \rho^6 \,. \end{aligned}$$

This is an exact, supersymmetric solution to the type IIB equations. The background metric exhibits a scaling symmetry under

$$x^i \mapsto a\,x^i\,, \qquad t \mapsto a^7\,t\,, \qquad \rho \mapsto a^{-1}\,\rho\,, \qquad x \mapsto a^2\,x\,, \tag{4.13}$$

which is an anisotropic Lifshitz-like scaling, with dynamical critical exponent $z = 7$, in the 4D gauge theory. Note that the scaling symmetry is not exact, as it is broken by the logarithmic running of the dilaton with energy scale.

Let us point out certain noteworthy aspects of the scaling solution (4.12). The IR Lifshitz scaling is realized in an unusual fashion due to the non-trivial dependence of the metric on two holographic directions, x and ρ, and this will further affect the scaling properties of observables such as the entanglement entropy and thermodynamic quantities. It is remarkable that the $z = 7$ Lifshitz scaling, with exactly the same running of the dilaton, was also observed in [72, 73] within the non-supersymmetric $SO(6)$-symmetric ansatz for smeared Wilson lines in $\mathcal{N} = 4$ SYM. This suggests that the emergent Lifshitz symmetry with $z = 7$ may be a universal feature of a dense state of heavy quarks introduced in $\mathcal{N} = 4$ SYM, independent of the details of their internal $SO(6)$ orientation. A further difference between the $SO(6)$-symmetric setup of [72, 73] and the supersymmetric configuration above, is that whilst the former has $F_3 \neq 0$ corresponding to D5-brane/baryon charge density, the latter has $F_3 = 0$. However this detail does not appear to affect the dynamical critical exponent in the IR. A supersymmetric configuration with $F_3 \neq 0$ is explored in chapter 5.

We can recover a more conventional form of the Lifshitz-like metric by taking a further limit of the IR solution, namely $x^2\rho^4 \ll 1$. After some coordinate rescalings, this yields

$$\begin{aligned} \mathrm{d}s^2_{\text{Einstein}} &\approx c\left(\frac{\mathrm{d}\rho^2}{\rho^2} - \rho^{14}\mathrm{d}t^2 + \rho^2\sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i + \rho^4\mathrm{d}x^2 + \tfrac{1}{16}\mathrm{d}\Omega_4^2\right) \\ \mathrm{e}^{\phi} &= \rho^6\,, \end{aligned} \tag{4.14}$$

where c is a constant. The four-sphere has a constant size in this limit, and the background (4.14) is not, by itself, an exact solution to the supergravity equations of motion; it is a limiting form of the IR (small y) metric. An interesting aspect of the IR physics that is manifest in this limit, is that it is effectively a six-dimensional geometry (after reduction on S^4). In this sense the situation is reminiscent of certain Coulomb branch configurations of $\mathcal{N} = 4$ SYM that deconstruct a higher dimensional field theory [90]. We will return to this picture in the next section when we discuss smeared strings on the Coulomb branch.

Entanglement entropy: An important probe of the IR physics of the gauge theory is the entanglement entropy, which can be computed using the Ryu-Takayanagi prescription [91]. For the metric in (4.12), this is not a straightforward calculation since the components depend both on x and ρ coordinates and the solution for the extremal surface (with a prescribed boundary) could depend non-trivially on both these dimensions.

We can, however, make a simplifying approximation by assuming that the physics in the deep infrared should be governed by sufficiently small ρ and x, so that $x^2\rho^4 \ll 1$. Then we can use the approximate form (4.14) to calculate the entanglement entropy of a 'strip' in $\mathbb{R}^3$, specified by

$$-\ell \leq x^1 \leq \ell\,, \qquad 0 \leq x^{2,3} \leq L\,, \qquad 0 \leq x \leq \tilde{L}\,, \tag{4.15}$$

with $\ell \ll L, \tilde{L}$ at a UV-slice of the geometry, $\rho = \rho_\Lambda$. The effective area functional for the 3D surface Σ_3 with the strip as its boundary, can be defined as [91]

$$\mathcal{S} = \frac{1}{4G_N} \int_{\Sigma_3 \times S^4 \times \mathbb{R}_x} \mathrm{d}^8x \sqrt{\det {}^*g}\,. \tag{4.16}$$

Extremizing the functional and extracting its finite part in the usual way (e.g. [5, 69]) we find

$$\mathcal{S}\Big|_{\text{finite}} \sim N^2 \tilde{L} \left(\frac{L}{\ell}\right)^2 \ell^{-2}\,, \tag{4.17}$$

where we have omitted various constants of proportionality and traded Newton's constant in ten dimensions G_N for a factor of N^2 according to the holographic dictionary. The scaling of the entanglement entropy with ℓ is characteristic of hyperscaling violation with negative θ [5, 69]. If we view the (IR) gauge theory as having 3 spatial dimensions (e.g. with the x-coordinate compactified), then $\theta = -2$.

The hyperscaling violation could also be directly inferred by reducing the ten dimensional metric (4.14) to five dimensions, by assuming the x-coordinate to be compact. The reduction to five dimensions yields

$$\mathrm{d}s_5^2 = \rho^{4/3} \left(\frac{\mathrm{d}\rho^2}{\rho^2} - \rho^{14}\mathrm{d}t^2 + \rho^2 \mathrm{d}\vec{x}^2 \right)\,, \tag{4.18}$$

which exhibits Lifshitz scaling with $z = 7$ and hyperscaling violation with $\theta = -2$. For the configurations on the Coulomb branch studied below, the emergence of the extra spatial coordinate x can be interpreted via deconstruction and the size of this dimension is controlled by the inverse spacing between D3-branes [90, 92].

We stress that the calculation of the entanglement entropy outlined above is only an approximation. It would be interesting to have a more precise computation in the exact scaling background (4.12), and to check whether these qualitative expectations are reproduced.

4.2.4 UV AdS asymptotics

We will now attempt to understand how the solution to (4.6) will modify the UV AdS asymptotics. In order to find a solution which asymptotes to $\mathrm{AdS}_5 \times \mathrm{S}^5$, and also includes string sources, we need to use the general form (4.7) for $h_1(y)$. Whilst the problem is not analytically tractable, we can make progress by noting that (4.6) exhibits translational invariance in x. Fourier transforming with respect to this variable [93] yields

$$\tilde{h}_3''(y;p) + \frac{4}{y}\,\tilde{h}_3'(y;p) - p^2\, h_1(y)\,\tilde{h}_3(y;p) = 0\,, \qquad h_1(y) = 1 + \frac{Q_1}{y^3}\,, \tag{4.19}$$

where primes denote derivatives with respect to y. The equation has irregular singular points of order 1/2 at $y = 0$ and order 1 at $y = \infty$. Around either of these points, the equation can be solved as a power series using a so-called Thomé expansion [94]. Around $y = \infty$, this leads to solutions of the type,

$$\tilde{h}_3(y;p) = \frac{\mathrm{e}^{py}}{y^2} \sum_{n=0}^{\infty} \frac{a_n(p)}{y^n}\,, \tag{4.20}$$

where the coefficients can be determined via a recursion relation and p can be either positive or negative. A slightly different expansion which has overlap with the large y limit is a formal series expansion in powers of Q_1 – the string or 'heavy quark' density:

$$\tilde{h}_3(y;p) = \sum_{n=0}^{\infty} Q_1^n\, f_n(y,p)\,. \tag{4.21}$$

Substituting into (4.6), and solving the resulting equations order by order in Q_1, we find,

$$f_0(y,p) = \frac{\mathrm{e}^{py}}{y^2}\left(1 - \frac{1}{py}\right)\,, \qquad f_1(y,p) = -\frac{p\,\mathrm{e}^{py}}{4\,y^4}\,, \tag{4.22}$$

$$f_2(y,p) = -\frac{p^5\,\mathrm{e}^{py}}{40y^3}\left(1 + \frac{3}{2py} + \frac{1}{p^2y^2}\right) + \frac{p^6\,\mathrm{e}^{-py}}{20y^2}\,\mathrm{Ei}(2py)\left(1 + \frac{1}{py}\right)\,,$$

for the first few terms in the expansion (p can be either positive or negative). $\mathrm{Ei}(x)$ is the exponential integral function. The correct superposition of the Fourier-transformed solutions must reproduce $\mathrm{AdS}_5 \times \mathrm{S}^5$ in the limit $Q_1 \to 0$. This picks out the required combination,

$$h_3(x,y) = \int \mathrm{d}p\,\frac{\mathrm{e}^{ipx}}{4}\left[\theta(p)\,p\,\tilde{h}_3(y;-p) - p\,\theta(-p)\,\tilde{h}_3(y;p)\right]\,. \tag{4.23}$$

In terms of the standard radial coordinate of AdS and the polar angle θ on the S^5 ($x = r\cos\theta$, $y = r\sin\theta$) to linear order in Q_1, we have,

$$h_3(r,\theta) = \frac{1}{r^4}\left(1 + \frac{Q_1}{r^3}\frac{1-4\cos^2\theta}{4\sin^3\theta} + \mathcal{O}(Q_1^2)\right) . \tag{4.24}$$

The dilaton, on the other hand, is exactly determined by

$$e^{\phi} = h_1^{-1/2} = \left(1 + \frac{Q_1}{r^3\sin^3\theta}\right)^{-1/2} , \tag{4.25}$$

and vanishes at $\theta = 0, \pi$ i.e. the north and south poles of the S^5 in the UV geometry, as expected for F1-string sources localized at $y = 0$. It is instructive to look at the corrections to the metric components in the above expansion,

$$g_{ii} = r^2 + \frac{Q_1}{r}\frac{1+4\cos^2\theta}{8\sin^3\theta} + \mathcal{O}(Q_1^2) , \tag{4.26}$$

with similar corrections appearing for g_{tt}. The $1/r$ corrections to AdS asymptotics are characteristic of the backreaction due to string sources, as also noted in [72, 95, 96]. A similar analysis for the internal directions suggests that upon inclusion of the backreaction from the strings, the S^4 does not shrink at $\theta = 0, \pi$, and that there are curvature singularities at these points. We should stress, however, that we cannot trust the expansion in Q_1 in the vicinity of the string sources at $\theta = 0, \pi$. Away from these points (fixed generic θ and large r) the geometry is asymptotically, locally $\mathrm{AdS}_5 \times \mathrm{S}^5$.

To summarize, we have argued in this section that the introduction of an $\mathcal{O}(N^2)$ density of heavy quarks (specifically, mutually BPS quark-antiquark pairs), preserving some supersymmetry in $\mathcal{N} = 4$ SYM at large-N and strong coupling, induces a flow from $\mathrm{AdS}_5 \times \mathrm{S}^5$ to a scaling Lifshitz-like IR background with $z = 7$. The anisotropic scaling solution in the IR is somewhat novel due to the presence of *two* non-compact holographic directions, which scale differently to ensure that the resulting metric is scale invariant.

4.3 Coulomb branch solutions and hyperscaling violation

A notable feature of the supersymmetric intersections discussed above is that they inherit the Coulomb branch moduli space of the $\mathcal{N} = 4$ theory (where $SU(N)$ is Higgsed to $U(1)^{N-1}$ generically). In particular a probe D3-brane (spanning t, x^1, x^2, x^3) placed at any point in the x-y plane, in the general background (4.2), experiences no force due to an exact cancellation between the Dirac-Born-Infeld (DBI) and Wess-Zumino terms in the probe action. A similar cancellation occurs (between the Nambu-Goto action and the coupling to B_2) for a probe F-string oriented parallel to the x-axis at any value of the y-coordinate. This points towards the existence of more general smeared solutions where both D3-branes and the F-strings are smeared with some distribution functions on the x-y plane. We show that for certain choices of such distribution functions the backreacted (IR) geometries can exhibit a range of Lifshitz-like scalings with hyperscaling violation.

General solutions with D3-branes on the Coulomb branch (within our $SO(5)$-symmetric ansatz) can be obtained by considering a general D3-brane density, so that

$$h_3(x, y) = \int\int \mathrm{d}x'\,\mathrm{d}y' \frac{\rho_{\mathrm{D3}}(x', y')}{[(x-x')^2 + (y-y')^2]^2} \,. \tag{4.27}$$

A general density function $\rho_{\mathrm{D3}}(x, y)$ preserves all supersymmetries whilst also leading to a violation of the Bianchi identity for F_5 due to the extended source distribution.

Since we want to interpret our solutions below as IR limits of Coulomb branch distributions it is useful to illustrate this with a simple example which allows to make contact with the F1-D3 intersection of [79]. Let us first consider a uniform distribution of D3-branes, in an interval along the x-axis with

$$\rho_{\mathrm{D3}}(x, y) = \rho_0\, \delta(y)\,, \qquad\qquad 0 \le |x| \le \tfrac{1}{2}\rho_0\,, \tag{4.28}$$

and no macroscopic string sources. For large (x, y) the geometry asymptotes to $\mathrm{AdS}_5 \times \mathrm{S}^5$, whilst in the IR, for small y and $|x| < \frac{1}{2}\rho_0$, we obtain

$$h_3 \simeq \frac{\pi}{2y^3}\,\rho_0 + \dots. \tag{4.29}$$

This is a scaling regime where all metric components in (4.2) only depend on powers of y. Upon performing a reduction of (4.2) on S^4 and the x-coordinate, the resulting 5D geometry precisely matches the reduction on a torus of the $\mathrm{AdS}_7 \times \mathrm{S}^4$ solution in 11D SUGRA [90]. This $SO(5)$ symmetric Coulomb branch configuration can therefore be viewed as a flow at strong coupling and large-N, to an IR theory which appears to be (a subsector of) the 6D superconformal theory with $(2, 0)$ supersymmetry realized on a stack of M5-branes (compactified on a two-torus). The Coulomb branch configuration therefore deconstructs a higher dimensional field theory. In the deconstruction picture, the sizes of the extra dimensions are controlled by the inverse spacings between individual D3-branes on the Coulomb branch. The spacings $\sim \mathcal{O}(1/N)$ determine the masses of the lightest W-bosons and dyonic states on the Coulomb branch, which in turn are related to the Kaluza-Klein harmonics of the deconstructed compact dimensions. We would now like to understand how the IR dynamics on the Coulomb branch is modified by smeared, macroscopic fundamental strings viewed as heavy quarks in the gauge theory.

4.3.1 Smeared F1-D3 intersections

In the presence of macroscopic string sources, both h_1 and h_3 will be non-trivial and the Bianchi identity for F_5 modified by D3-brane sources becomes

$$\partial_y \left(y^4 \partial_y h_3\right) + y^4 \, h_1 \partial_x^2 h_3 = -\rho_{\mathrm{D3}}(x, y) \,. \tag{4.30}$$

We will focus attention on distributions that, at least in some limit as in the above example, only depend on y, so that $\rho_{\mathrm{D3}} = \rho_{\mathrm{D3}}(y)$. Hence, for simplicity, we also assume that h_3 is a function of y alone. Similarly the equation of motion for H_3 with a general smeared distribution of F-strings leads to the equation

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(y^4 \frac{\mathrm{d}h_1}{\mathrm{d}y}\right) = -\rho_{\mathrm{F1}}(y). \tag{4.31}$$

As pointed out above, probe F-strings experience no force when they are aligned along the x-axis and placed at any value of y. Therefore a general y-dependent distribution should preserve 1/4-supersymmetry. In the Appendix we show that such smearing of sources satisfies a calibration condition which ensures that the string/brane embeddings we study respect the supersymmetries. The equations of motion follow from the type IIB supergravity action coupled to the smeared D3-brane and string sources,

$$S = S_{\mathrm{IIB}} + S_{\mathrm{NG}} + S_{\mathrm{DBI}} + \frac{1}{2\pi\alpha'} \int B_2 \wedge \Omega_8 + 4 T_{\mathrm{D3}} \int C_4 \wedge \Omega_6. \tag{4.32}$$

The actions for the sources are the smeared versions of the Nambu-Goto and DBI actions with a particular choice of smearing forms Ω_8 and Ω_6, respectively

$$\Omega_8 = -\frac{\pi\alpha'}{\kappa^2} \rho_{\mathrm{F1}}(y) \; \mathrm{d}y \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \epsilon_{(4)} \tag{4.33}$$

$$\Omega_6 = -(2\kappa^2 T_{\mathrm{D3}})^{-1} \; \rho_{\mathrm{D3}}(y) \; \mathrm{d}x \wedge \mathrm{d}y \wedge \epsilon_{(4)} \,, \tag{4.34}$$

where we have defined $2\kappa^2 \equiv 16\pi G_N$. The sources alter the equations of motion for the dilaton, H_3, F_5 and the metric. In particular they contribute to the stress tensor. Imposing the weak energy condition (WEC) on the source stress tensor leads to positivity of the source density functions

$$T_{AB}u^A u^B \geq 0\,, \qquad \Longrightarrow \qquad \rho_{\rm D3}\,, \rho_{\rm F1} \geq 0\,, \tag{4.35}$$

where u_A is some timelike vector. For more details on the equations of motion and their consistency with supersymmetry, we refer the reader to appendix D.

To obtain scaling solutions, we simply choose power laws for the smearing densities

$$\rho_{\rm F1} = \alpha\,(3-\alpha)\; Q_1\, y^{2-\alpha}\,, \qquad\qquad \rho_{\rm D3} = \beta\,(3-\beta)\; Q_3\, y^{2-\beta}\,. \tag{4.36}$$

Then the equations of motion for h_1 and h_3 are also solved by power laws

$$h_1 = \frac{Q_1}{y^\alpha}\,, \qquad\qquad h_3 = \frac{Q_3}{y^\beta}\,. \tag{4.37}$$

Positivity of the source densities requires $0 \leq \alpha, \beta \leq 3$. Further imposing the null energy condition (NEC) on the complete stress tensor leads to a weaker condition on the parameters α, β which is consistent with the WEC . The case $\alpha = \beta = 3$ gives us the homogeneous solution of Dey and Roy [79]. This is the situation wherein the D3-branes and strings are all at $y = 0$, but the branes are uniformly smeared along the x-axis.

Inserting the general scaling solution into the metric (4.2) and reducing carefully over the S^4 and the x-coordinate (which we treat as a deconstructed compact dimension) to five-dimensional Einstein frame we obtain a family of Lifshitz geometries with hyperscaling violation:

$$\mathrm{d}s_5^2 = R^2\, r^{-\frac{2}{3}\theta} \left(-r^{2z}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + r^2 \sum_{i=1}^{3} \mathrm{d}x^i \mathrm{d}x^i \right)\,. \tag{4.38}$$

Here the radius is $R^2 = \frac{4}{(\beta-2)^2} Q_3^{\frac{1}{2}} Q_1^{\frac{1}{4}}$ and we have changed the radial coordinate according to $y = r^{\frac{2}{\beta-2}}$ and rescaled the rest. The coefficients are related to the exponents of the sources as

$$z = 1 + \frac{\alpha}{\beta - 2}\,, \qquad\qquad \theta = \frac{\alpha + 4\beta - 14}{\beta - 2} \tag{4.39}$$

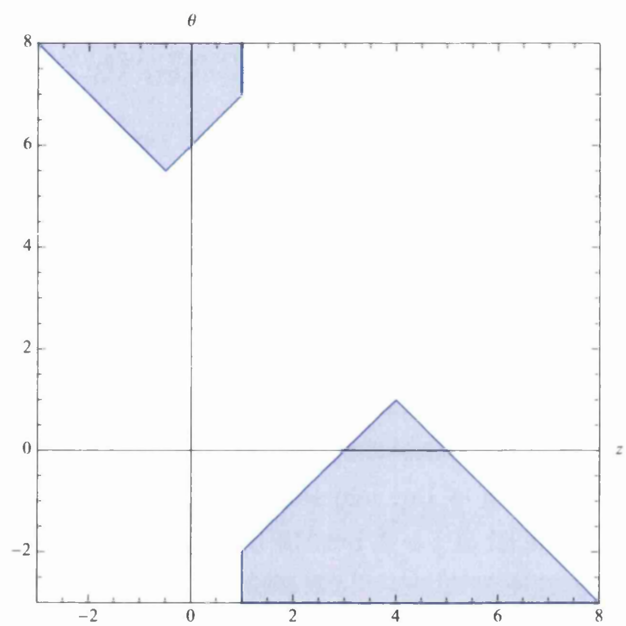

Figure 4.2: The shaded region represents the allowed values for the dynamical exponent z and hyperscaling violation coefficient θ for the solutions discussed in the text. If we impose an additional requirement that $\theta \leq d$ for stability (e.g. [5]), this would also exclude the shaded region top-left corner (which has $\beta < 2$).

The case $\beta = 2$ needs to be treated separately, as we do below. Given that the weak energy conditions for the sources, considered separately, restrict the values of the parameters to lie in the range $0 \leq \alpha,\ \beta \leq 3$, the allowed values for z and θ in this class of solutions are shown in figure 4.2. Note that if we also require $\theta \leq d$ for stability as argued in [5], only the solutions shown in the lower right corner of figure 4.2 with $\theta \leq 1$ survive. These correspond to D3-brane distributions with $\beta > 2$. Finally, there are no solutions with $\theta = 2$, which would lead to a logarithmic violation of the area law for entanglement entropy.

$\beta = 2$: **Conformally** $\mathrm{AdS}_2 \times \mathbb{R}^3$: As stated before, the case $\beta = 2$ has to be treated separately, since the change of variables used for the radial coordinate is not well defined for that value of the parameter. Importantly as $\beta \to 2$, both z and θ diverge,

$$\lim_{\beta\to 2} \quad z, \theta \to \infty \qquad\qquad \lim_{\beta\to 2} \quad \eta \equiv -\tfrac{\theta}{z} \to \left(\tfrac{6}{\alpha} - 1\right) . \tag{4.40}$$

In the allowed range for α, we have $\eta \geq 1$. The $\beta \to 2$ limit is also interesting because it corresponds to a uniform distribution of D3-branes on the x-y plane,

$$\rho_{\mathrm{D3}} \ = \ 2\,Q_3\,, \tag{4.41}$$

so that the harmonic function is,

$$h_3 \ = \ \frac{Q_3}{y^2}\,. \tag{4.42}$$

and $h_1 = Q_1/y^\alpha$ as before. Hence we obtain a one-parameter ($\alpha \neq 0$) family of solutions. Again, substituting this into the ten-dimensional metric and reducing to the five-dimensional Einstein frame with the change of variable $y = r^{\frac{2}{\alpha}}$ we arrive at a metric conformal to $\mathrm{AdS}_2 \times \mathbb{R}^3$,

$$\mathrm{d}s_5^2 \ = \ R^2\, r^{\frac{2}{3}\eta} \left(-r^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \sum_{i=1}^{3} \mathrm{d}x^i \mathrm{d}x^i \right) , \tag{4.43}$$

where the radius is now $R^2 = 4\alpha^{-2} Q_3^{\frac{1}{2}} Q_1^{\frac{1}{4}}$. Geometries with an IR factor conformal to $\mathrm{AdS}_2 \times \mathbb{R}^2$ have been used to describe locally quantum critical theories and argued to have certain properties desirable for a holographic description of Fermi surfaces [71]. The zero temperature entropy density associated to the conformally $\mathrm{AdS}_2 \times \mathbb{R}^3$ background is vanishing since $\eta > 0$ and the entanglement entropy has no extensive finite contribution. As pointed out in [71], the entanglement entropy of a strip in this case has a rather puzzling behaviour in that the extremal surface only exists for a specific value of the strip width.

In all the geometries, including the conformally $\mathrm{AdS}_2 \times \mathbb{R}^3$ backgrounds, curvature singularities appear in the deep IR (see e.g. [97]). In particular, since the dilaton is given by $e^{\phi} = h_1^{-1/2}$, and runs to zero in the IR, the string frame metric leads to divergent curvatures and the solutions will be expected to receive large α' corrections. One can S-dualize the backgrounds, and then the problem becomes one of large string coupling in the IR. It is a compelling question to ask what low energy physics arises and resolves such curvature singularities [98, 99]. This may be an interesting avenue to explore since we have some knowledge of the microscopic field theory the geometries describe.

4.4 Discussion

The supersymmetric scaling geometries we have discussed in this chapter were obtained from previously known intersecting brane configurations along with a new ingredient, namely, generic source distributions compatible with the supersymmetries. Our motivation was to understand whether any universal features emerge within a tractable holographic framework, when a state of finite (quark) density is introduced into a known field theory (with a large-N string/gravity dual). The physical interpretation of the setup explored in this paper is that it corresponds to a uniform density of mutually BPS quarks and anti-quarks (with opposite $SO(6)$ orientations) in $\mathcal{N} = 4$ SYM. In particular, these configurations all have vanishing three form flux, F_3. In the next chapter we discuss solutions with non-zero F_3, which can be interpreted as a finite baryon density, as was the case in [72] (see also [100] for related discussions).

One of the larger aims of this study is to understand whether the simplistic picture of backreacting smeared heavy quarks can be embedded into holographic setups with dynamical flavours, most notably the smeared D3-D7 system explored, for instance, in [83–85].

An important observation is the emergence of a $z = 7$ IR scaling regime from the partially localized intersection described by homogeneous (sourceless) equations. Despite having different global symmetries and no supersymmetry, the same scaling was observed in [72, 73]. The value of z can now be understood within the intersecting brane framework. Following the results of [78], it is easy to verify that, more generally, partially localized F1-Dp intersections give rise to anisotropic scaling with $z = \frac{16-3p}{4-p}$, when $p < 4$.

Finding finite temperature, black brane generalizations of the solutions discussed in this paper appears difficult. For the partially localized intersection (with $z = 7$), the solutions are functions of *two* coordinates, $v = x^2 y$ and y, and this makes the problem of 'blackening' the solutions challenging[1]. We leave this question for future investigation. When the sources are smeared, as for the Coulomb branch configurations, the zero temperature delocalization is possible only because the theory has a moduli space of vacua. Finite temperature lifts such moduli spaces so that the free energy is minimized at the origin of moduli space[2]. A possible approach towards the finite temperature physics of smeared Coulomb branch distributions is to introduce chemical potential for charges under the global symmetry group, i.e. to consider rotating brane configurations along the lines of [102, 103].

Within the context of the Coulomb branch intersctions, it is interesting to note that by engineering appropriate string/D3-brane sources we should be able to obtain zero temperature flows between different scaling geometries.

[1] See however [101]

[2] Away from the origin, various degrees of freedom are rendered massive, and their contribution to the entropy thus decreases.

Chapter 5

Smeared baryon vertices

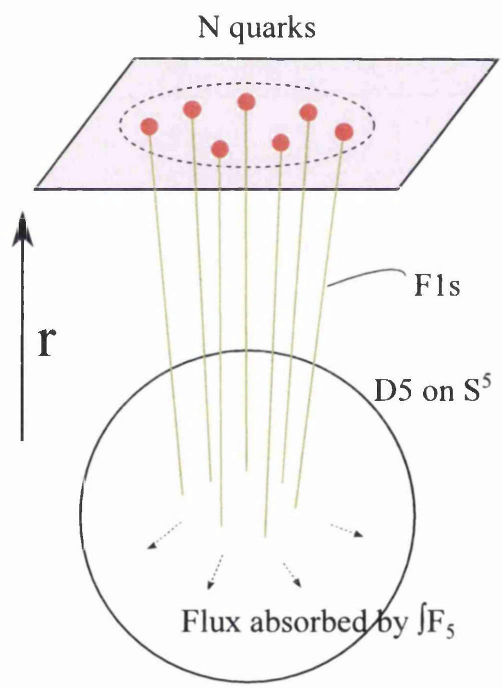

Figure 5.1: The baryon vertex in $\mathcal{N} = 4$ SYM

In this chapter we consider the same smeared string supergravity ansatz as in chapter 4, but now with non-zero RR three-form flux F_3. This flux is magnetically sourced by D5-branes, which provide the baryon vertex construction of [104], in which N strings end on a D5 brane wrapping S^5. The Gauss law on the compact space is satisfied since the WZ coupling

$$S_{WZ} \sim \int A \wedge F_5$$

couples the five-form as a charge for the gauge field. This pictured in figure 5.1. It is natural to interpret these solutions as being dual to a finite density of N-quark baryon states.

5.1 Solutions

The metric and fluxes take same form (4.2), as do the SUSY projections (4.3). The supergravity fields are now sourced by D3, D5 branes and fundamental strings, which are introduced through the standard combined DBI and WZ action and are fully back-reacted.

The functions A and ϕ are not arbitrary, but are determined by the consistent coupling to these sources, which enter into the Einstein equations and flux equations of motion. We look for solutions which have at most localized (delta function) sources for D3 and D5 branes, but we will allow for a smeared distribution of fundamental strings. Inspection of the equations of motion for the system shows that the branes must have the configuration

	t	y	x	x^1	x^2	x^3	θ^1	θ^2	θ^3	θ^4
$D3$	$\times$			$\times$	$\times$	$\times$				
$D5$	$\times$	$\times$					$\times$	$\times$	$\times$	$\times$
$F1$	$\times$		$\times$							

The kappa symmetry conditions for these brane orientations are consistent with the SUSY projectors.

The form equations of motion and Bianchi identities are given by

$$\begin{array}{cc} \mathrm{d}(\mathrm{e}^{-\phi} * H_3) + 4F_5 \wedge F_3 = \Omega_8 & \mathrm{d}(\mathrm{e}^{\phi} * F_3) + 4H_3 \wedge F_5 = 0 \\ \mathrm{d}F_3 = 0 & \mathrm{d}F_5 - \frac{1}{4}H_3 \wedge F_3 = 0 \end{array}$$

where Ω_8 is the (appropriately normalized) smearing form for the F1 distribution. We will consider the case in which $F_3 \neq 0$; the $F_3 = 0$ was treated in chapter 4. As shown in section 3.4, the endpoint of the analysis is that the whole system is governed by a non-linear Poisson-type equation

$$\frac{1}{y^4}\partial_y(y^4\partial_y \mathrm{e}^{-2\phi}) + \frac{1}{2}\partial_x^2 \mathrm{e}^{-4\phi} = \rho(y) \quad . \tag{5.1}$$

A is expressed in terms of the dilaton through $\mathrm{e}^{-4A} = -(2/F_0)\partial_x \mathrm{e}^{-\phi}$, where F_0 is a constant. This solves the above equations of motion as well as the Einstein and dilaton equations with source terms for the F1s, when we identify $\Omega_8 = y^4\rho(y)\mathrm{d}y \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge vol(S^4)$. It is similar to the Toda equation which appears in the analysis of [35, 105]. In that case there was an implicit variable change which mapped the problem into a linear electrostatics problem. However it seems to us unlikely that a similar variable change can be made in our case.

5.2 UV perturbation about $AdS_5 \times S^5$

Taking $\mathrm{e}^{\phi} = 1$, we recover vacuum AdS, in the coordinates (4.5). If we linearize around this, $\mathrm{e}^{-2\phi} = 1 + \epsilon\, h_{(1)} + O(\epsilon^2)$, we find that $h_{(1)}$ satisfies the $SO(5)$ symmetric Laplace equation on flat $\mathbb{R}^6$:

$$\frac{1}{y^4}\partial_y(y^4\partial_y h_{(1)}) + \partial_x^2 h_{(1)} = 0 \tag{5.2}$$

Imposing AdS asymptotics, we find a family of solutions of this Laplacian which as usual takes the form of sum over point charges

$$h_{(1)} = \frac{1}{y^2}\sum_i Q_i\left[\frac{(x-x_i)}{(x-x_i)^2+y^2} + \frac{1}{y}\left(\arctan\frac{x-x_i}{y} + \frac{\pi}{4}\right)\right] \tag{5.3}$$

As a simple example we can take two charges $Q_{1,2} = +1$ at $x_1 = -x_0$, $x_2 = +x_0$. In order to normlize the AdS correctly, we are forced to make the identification $F_0 = -2\epsilon$, which implies that the ϵ expansion is equivalent to an expansion in F_0. Then we find the following asymptotic metric in polar coordinates:

$$h_{ii} = r^2 - x_0^2(2+3\cos 2\theta) - \frac{F_0}{16}\frac{1}{r}\frac{1}{\sin^3\theta}\left(2(\pi-\theta)+\sin 2\theta\right) + O(r^{-4}, F_0^2) \tag{5.4}$$

This displays the $1/r$ potential term typical of backreacted string sources, and also the expected term corresponding to vevs for the adjoint scalars of the field theory. The dilaton is

$$\mathrm{e}^{-2\phi} = 1 - \frac{F_0}{2}\frac{2(\pi-\theta)+\sin 2\theta}{\sin^3\theta}\frac{1}{r^3} + O(F_0^2) \tag{5.5}$$

and to this order in F_0 the S^5 is the same as for the vacuum $AdS_5 \times S^5$.

5.3 Scaling solution

It is interesting to observe that (5.1) has an exact solution of the form $e^{-2\phi} = \frac{1}{y^3}\alpha(p)$, where $p \equiv x^2 y$. Indeed when we substitute in this ansatz, the p-dependence decouples and we are left with

$$p(4\alpha + p)\alpha'' + 4\,p\,\alpha'^2 + 2(\alpha - p)\alpha' = 0 \tag{5.6}$$

In particular we have been able to find at least one closed form solution to this equation, by separation of variables: $\alpha(p) = \frac{p}{3}$.

We can again find the function $f = \frac{1}{F_0}(\frac{n_{F1}}{4V_{S^4}} + \frac{x^2 y}{6})$ whose difference along a curve counts the number of $D3$-branes contained within that curve. It is only a function of p, so that the $D3$-branes are 'delocalized' in the same sense as in section 4.2.3. The constant n_{F1} tells us something about the number density of fundamental strings, as a calculation of the $F1$ Page charge shows:

$$N_{F1}^{\text{Page}} \equiv \int_{\mathbb{R}^3 \times S^4} \left(e^{-\phi} * H_3 + 4\,F_3 \wedge C_4\right) = n_{F1} V_{\mathbb{R}^3}$$

However since we are free to choose n_{F1} without changing the flux F_5, N_{F1}^{Page} does not seem to tell us any invariant information. This is to be expected since the Page charge is not gauge invariant [106]. There may however be some 'preferred' choice of gauge which fixes n_{F1} on physical grounds. Such an argument might come from the D1-brane Page charge

$$\begin{aligned} Q_{D1} &\equiv \int_{\Sigma_7} \left(e^{\phi} * F_3 - 4\,H_3 \wedge C_4\right) && (5.7) \\ &= \frac{4}{F_0} \int_{\Sigma_7} \left(F_0\, n\, e^{2\phi} \partial_y \phi + y^4((\partial_y \phi)^2 + e^{-2\phi}(\partial_x \phi)^2)\right) dt \wedge dx \wedge dy \wedge vol(S^4) && (5.8) \end{aligned}$$

which is also gauge dependent, since, again, n is a constant related to the gauge of C_4.

On the other hand the constant F_0, which tells us the number of units of F_3 flux which are needed to satisfy Gauss' law at the endpoint of the F1-string, is gauge invariant and should also do the job of counting the number of strings.

What we want to stress is that for all solutions solving (5.6), the metric is invariant under a $z = 7$ Lifshitz scaling ($x \mapsto ax$, $y \mapsto a^{-2}y$, $t \mapsto a^7 t$, $x^i \mapsto ax^i$) in which both 'holographic' directions scale non-trivially. The scaling can be made most transparent if we change to new coordinates (ρ, σ):

$$\begin{aligned}
\mathrm{d}s^2 &= \alpha^2 Q \left(-\rho^{10}\,\sigma^4 \mathrm{d}t^2 + \rho^2\,\mathrm{d}x^i \mathrm{d}x^i + \frac{\mathrm{d}\rho^2}{\sigma^2} + \frac{3}{4}\frac{\rho^2}{\sigma^4}\,\mathrm{d}\sigma^2 + \frac{1}{16}\frac{\rho^2}{\sigma^2}\mathrm{d}\Omega_4^2 \right) \\
F_5 &= \frac{\alpha^6}{128}\,Q^2\,(1+*)\,\mathrm{d}\left(\frac{\rho^4}{\sigma^4}\right) \wedge vol(S^4) \\
H_3 &= \frac{3}{2}\,Q\,\rho^6 \sigma\,\mathrm{d}t \wedge \mathrm{d}\rho \wedge \mathrm{d}\sigma \\
F_3 &= Q\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\mathrm{e}^{\phi} &= \rho^4 \sigma^2
\end{aligned}$$

where $\alpha \equiv \frac{3^{1/4}}{2^{3/2}}$, and we defined $Q \equiv -F_0$ (we see here that F_0 must be negative for the metric to have the right signature). Now the symmetry is realized as

$$x^i \mapsto a\,x^i\,, \qquad t \mapsto a^7\,t\,, \qquad \rho \mapsto a^{-1}\,\rho\,, \qquad \sigma \mapsto a^{-1}\,\sigma$$

This metric has vanishing Ricci scalar $R = 0$. We also have the curvature invariant

$$R^{MN}R_{MN} = \frac{29536}{3\alpha^2}\frac{1}{Q^2}\frac{\sigma^4}{\rho^4} \tag{5.9}$$

Therefore there is a curvature singularity situated right the way along the σ-axis. This may be related to the F1 string which could be regarded as lying there.

It therefore seems that a non-zero baryon density, which should be interpreted as confinement of the probe quarks, does not change the fact that we have a Lifshitz scaling. When taken with the results of [72], $z = 7$ scaling therefore seems to be a robust feature of a finite density of heavy quarks in $\mathcal{N} = 4$ SYM which is indenpendent of supersymmetry, baryon density. Since we did not need to specify the $SU(N)$ representation of the Wilson lines, we may also be able to say the results are independent of this too. However without a detailed picture of a complete RG flow it is difficult to make this statement with complete confidence.

5.4 Discussion

Our setups are in the infinitely massive limit of the D3/D7 system, dual to $\mathcal{N} = 4$ SYM coupled to N_f hypermultiplets. This system has been most rigorously studied in the probe approximation, for both $(T, \mu) \neq 0$, with the rationale that in the 't Hooft limit we have $N_f \ll 2N$, and so the D7 brane backreaction is negligible. However at $(T = 0,\ \mu \neq 0)$, fluctuations of the probes are governed by a near-horizon $AdS_2 \times \mathbb{R}^3$ geometry. The fact that this has non-zero entropy violates the 3rd law, and suggests it is not the actual gound state. Analysis has however shown that it is thermodynamically stable [107]. Motivated by this, [108] searched for solitonic extrema of the D7 brane action. They found a Higgs branch-like moduli space of vacua, which is surprising since non-zero μ breaks all supersymmetry.

A possible resolution to these puzzles is suggested by the following simple argument: if the true vacuum has zero energy $E_0 = 0$, and there is a non-zero density of flavours which sets a scale $\rho^{1/4}$, we cannot be in a regime where $\rho^{1/4}/E_0 \ll 1$ (such that the ground state energy dominates over the energy of the flavours.) Therefore we should leave the probe approximation and include their backreaction.

The backreaction of flavours in this setting $(\mu, T \neq 0)$ has been considered [84], in the context of the Veneziano limit $N_f \sim N$ of the theory[1], perturbatively in N_f/N. However, perhaps unsurprisingly in light of the previous argument, these solutions cease to be valid as $T/\mu \to 0$.

We may view our own solutions as encoding the appropriate backreaction, at least in the large mass limit for the flavours. It seems that these backreacted geometries carry inside them a subsector (the metric and five-form flux) with $z = 7$ Lifshitz scaling symmetry. Since these are the fields representing the $SU(N)$ gauge sector, it is tempting to think of a picture where the 'flavour' degrees of freedom run in loops and cause the 'colour' degrees of freedom to become scale invariant. It is mysterious from the field theory viewpoint why there should be such an emergent symmetry. A natural question to ask is whether this effect is a feature specific to strong coupling, or whether it can be seen in the perturbative regime. If it could, then perhaps a mean-field analysis, along the lines of [109] but at finite density, would be illuminating.

<u>Lattices:</u>

Another perspective on our setup is to view it as the IR limit of an infinite lattice of string/D5 brane defects in $AdS_5 \times S^5$. This problem has been considered in [110], which argued that the gravitational warping means that backreaction of the lattice always becomes important in the IR because the covariantized density becomes large. They sought $AdS_2 \times \mathbb{R}^3$ backreacted solutions (which as we have seen is a reasonable

[1]Backreaction in the $\mu = 0$ case has also been considered. For a review see [85].

idea of what to expect at $T=0$), but to no avail. We suggest that our Lifshitz solutions also give the backreaction of these lattices.

$z=7$ Lifshitz:

Let us discuss the implications of the scaling symmetry. Lifshitz scaling was first discovered in field theory at critical points in certain condensed matter systems [61, 111]. It is a spacetime symmetry under which

$$t \mapsto a^z\, t \qquad x^i \mapsto a\, x^i$$

The simplest example is a free theory, the 'Lifshitz field theory', with $z=2$:

$$S = \int \mathrm{d}t\, \mathrm{d}^2x\ \left[(\partial_t\phi)^2 - \kappa(\partial_i\partial_i\phi)^2\right]$$

which has a line of fixed points parametrized by κ.

In order to attain such a symmetry, the effective action of such a theory must be such that the coefficients of all spatial derivative terms vanish up to $2z$ derivatives. For example, in the scalar field example above, the coefficient of $\partial_i\partial_i\phi$ must have vanished. As z increases, in the general case there will be more and more lower order spatial derivative terms which are allowed by the symmetries, all of which must vanish at the fixed point. It therefore seems as if a point with z as high as 7 is a very special one. Alternatively one may speculate that there is a symmetry which forbids all such lower derivative terms from appearing.

Appendix A

Notes on Clifford algebras and spinors

Start with the Clifford algebra in d spacetime dimensions:

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta_{\mu\nu}\,\mathbb{1}$$

We use $(-++\cdots)$ signature, and in this appendix γ^μ will refer to the gamma matrices themselves, as opposed to the 10D van der Waerden-type matrices used elsewhere in this thesis.

A.1 Even dimensions

We can recursively find a representation for the Clifford algebra in any even dimension. For example:

$$\underline{d=2}: \quad \gamma^0 = i\sigma^2 \quad \gamma^1 = \sigma^1$$

$$\begin{aligned} \underline{d=4}: \quad \gamma^{0,1} &= i\sigma^2, \sigma^1 \otimes \mathbb{1} \\ \gamma^{2,3} &= \sigma^3 \otimes \sigma^{1,3} \end{aligned}$$

This an irreducible representation of the *Clifford algebra.* Can show that this representation is *unique* up to a change of basis (similarity transformation, $\gamma^\mu \to M\gamma^\mu M^{-1}$).

Although the Dirac representation is irreducible as a representation of the Clifford algebra, it is *reducible* as a representation of the Lorentz algebra. This generated by the matrices:

$$\Sigma_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

One sees this by constructing the **chirality matrix**:

$$\Gamma \equiv i^\kappa \gamma^0 \gamma^1 \gamma^2 \cdots \gamma^{d-1}$$

where κ is either 0 or 1, which guarantees that $\Gamma^2 = \mathbb{1}$.

This matrix anti-commutes with all the γ^μ, so commutes with the $\Sigma_{\mu\nu}$. Since $\Gamma^2 = \mathbb{1}$, the eigenvalues of Γ are ± 1 ('chirality'). This divides the Dirac representation space into two subspaces invariant under the action of the Lorentz group. Therefore the Dirac representation is a reducible representation of the Lorentz group, consisting of a direct sum of two Weyl representations of opposite chirality.

A.2 Odd dimensions

In order to extend our Clifford algebra representation in the basis of section A.1 to the odd d of one dimension higher, we must an extra gamma matrix γ^{d-1}. We choose

$$\gamma^{d-1} = \pm\Gamma$$

i.e. the chirality matrix in one dimension down. The choice of sign actually defines two representations which are inequivalent (not related by a rotation of basis.) Therefore the representation is not uniquely defined up to similarity transformations: there are two inequivalent irreducible reps of the Clifford algebra. However now either of these are *irreducible* as representations of the Lorentz group. We can still form a 'chirality matrix' $\Gamma \propto \gamma^0 \cdots \gamma^{d-1}$ which squares to one and commutes commutes with the Lorentz generators. However now Γ *commutes* with all the γ^μ. Given the fact that either Dirac rep is an irreducible rep of the γ matrices, Schur's lemma tells us that any matrix which commutes with all the γ^μ must be a multiple of the identity matrix. So while Γ can be defined, it is trivial and does not define distinct subspaces. Another way of saying this is that although its eigenvalues are ± 1, it transpires that all of them are actually of the same sign.

A.3 Intertwiners

Note that $\pm\gamma^\mu$, $\pm(\gamma^\mu)^*$, $\pm(\gamma^\mu)^t$ and $\pm(\gamma^\mu)^\dagger$ all satisfy the Dirac algebra. In even dimensions, by the above uniqueness property, they are therefore related to the original γ's by a change of basis. Thus we have

$$\begin{aligned}
-\gamma^\mu &= \Gamma\gamma^\mu\Gamma^{-1} \\
\pm(\gamma^\mu)^* &= \mathcal{B}_\pm\gamma^\mu\mathcal{B}_\pm^{-1} \\
\pm(\gamma^\mu)^t &= \mathcal{C}_\pm\gamma^\mu\mathcal{C}_\pm^{-1} \\
\pm(\gamma^\mu)^\dagger &= \mathcal{A}_\pm\gamma^\mu\mathcal{A}_\pm^{-1}
\end{aligned}$$

$\{\mathcal{A}_\pm, \mathcal{B}_\pm, \mathcal{C}_\pm\} \equiv \{\mathcal{I}_\pm\}$ are called 'intertwiners' [112], since they relate different bases of similar representations. $\mathcal{B}$ (sometimes also referred to as $\mathcal{D}$) is often sometimes called the 'complex conjugation matrix' [23], and $\mathcal{C}$ the 'charge conjugation matrix'.

So in even dimensions, all of the intertwiners exist, and indeed we have the relations $\mathcal{I}_- = \Gamma\mathcal{I}_+$. In odd dimensions, $-\gamma^\mu$ and γ^μ are in opposite conjugacy classes, and so there is no interwiner between them. Thus only one of $\mathcal{I}_+, \mathcal{I}_-$ exists for each type of intertwiner. The three interwiners will always be related as $\mathcal{A} = \pm\mathcal{B}\mathcal{C}$. Let us take as an example the 3D basis

$$\gamma^0 = i\sigma^2 \quad \gamma^1 = \sigma^1 \quad \gamma^2 = \sigma^3$$

in which $\mathcal{B}_+ = 1$, $\mathcal{C}_- = \mathcal{A}_- = \sigma^2$.

A.4 Majorana spinors

The definition of the $\mathcal{B}$ intertwiner ensures that the 'complex conjugate' spinor $\psi^c \equiv \mathcal{B}^{-1}\psi^*$ transforms in the same way as ψ under Lorentz transformations. Therefore the condition

$$\psi = \psi^c \tag{A.1}$$

is Lorentz covariant. A spinor satisfying this condition is known a 'Majorana' spinor. The consistency condition for the existence of Majorana spinors, as can be seen from plugging A.1 back into itself, is that $\mathcal{B}^*\mathcal{B} = \mathbb{1}$. This is possible in $d = 2, 3, 4$, and again for $d = 8, 9, 10, 11$. In such dimensions, it may be possible to find a basis in which

$\mathcal{B} = 1$, so that the gamma matrices are either all real or all imaginary. This is known as a 'Majorana basis' - the Lorentz covariant conjugation reduces to simple complex conjugation, and a Majorana spinor is just a spinor with real components.

If the complex conjugation operation preserves the eigenvalue of Γ, we can simultaneously impose that the spinor is chiral and Majorana. In this case ψ is known as a 'Majorana-Weyl' spinor. This possible in $d = 2$ and $d = 10$.

A.5 Killing spinors (KS)

On round Euclidean spheres with coordinates $\{i\}$, Killing spinors satisfy:

$$\nabla_i \epsilon = \frac{i}{2R} \lambda \gamma_i \epsilon \tag{A.2}$$

where $\lambda = \pm$ and R is the radius. On even spheres, we have in addition

$$\nabla_i \epsilon = \frac{1}{2R} \lambda \gamma \gamma_i \epsilon \tag{A.3}$$

Where γ is the chirality matrix. On AdS spaces we have the same equation, with the AdS length L in place of R, and the factor of i now removed from (A.2) and inserted into (A.3).

A.5.1 Decomposing Killing spinors on S^{n+1} in terms of KS on S^n, for n even

In certain applications, it is useful to be able to decompose KS of higher dimensions spaces in terms those on lower dimensional ones. The example we want to consider is odd dimensional spheres in terms of the even dimensional sphere in one dimension lower, as dealt with an appendix of [22]. We wish to clarify the procedure as much as possible. We will set the sphere radii to unity.

For simplicity and explicitness we deal with S^3 in terms of S^2. We want to decompose the spinor on S^3 satisfying

$$\nabla_\mu \epsilon^3_+ = \frac{i}{2} \gamma_\mu \epsilon^3_a$$

in terms of the two spinors on S^2 satisfying the 'chiral' relation

$$\nabla_\mu \epsilon^2_\pm = \pm \frac{1}{2} \gamma \gamma_\mu \epsilon^3_\pm \quad .$$

We choose a basis of Killing spinors S^3: $\gamma^{\hat\psi} = \sigma^3$, $\gamma^{\hat\theta} = \gamma^{\hat 1} = \sigma^1$, $\gamma^{\hat\varphi} = \gamma^{\hat 2} = \sigma^2$, where on S^2, $\gamma \equiv \gamma^{\hat\psi}$ is the chirality matrix[1].

Explicit relations for these Killing spinors are found in [113]:

$$\epsilon^3_+ = e^{\frac{i}{2}\psi\gamma} e^{-\frac{1}{2}\theta\gamma_{\hat 2}\gamma} e^{-\frac{1}{2}\varphi\gamma_{\hat 1}\gamma_{\hat 2}}\epsilon_0 \tag{A.4}$$

$$\epsilon^2_\pm = e^{\pm\frac{1}{2}\theta\gamma\gamma_{\hat 2}} e^{-\frac{1}{2}\varphi\gamma_{\hat 1}\gamma_{\hat 2}}\epsilon_0 \tag{A.5}$$

where the ϵ_0's are arbitrary spinors. The space of $\epsilon^2_\pm$'s can be split into two pieces (call them $K_\pm$), with the projection $\gamma\epsilon_0 = \pm\epsilon_0$. The two spinors are related by on $K_\pm$

$$\epsilon^2_+ = \pm\gamma\epsilon^2_-$$

respectively. We deal with each in turn. From the expression above we can see that

$$\epsilon^3_+ = e^{\frac{i}{2}\psi\gamma}\epsilon^2_+ = \left(\cos(\frac{\psi}{2}) + i\sin(\frac{\psi}{2})\gamma\right)\epsilon^2_+ ,$$

so that using (A.5.1) on the two subspaces $K_\pm$ we can write the higher dimensional spinor as the linear combinations

$$\epsilon^3_+ = \cos\frac{\psi}{2}\epsilon^2_+ \pm i\sin\frac{\psi}{2}\epsilon^2_- .$$

Combining $K_\pm$ together again, we get back the same number of independent components as we expect.

We can also see this by decomposing the Killing spinor equations on S^3. First we write the metric of S^3 as

$$ds^2_{S^3} = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2)$$

then the spin connection is

$$\left(\omega^a{}_b\right) = \begin{pmatrix} 0 & -\cos\psi d\theta & -\cos\psi\sin\theta d\phi \\ \cos\psi d\theta & 0 & -\cos\theta d\phi \\ \cos\psi\sin\theta d\phi & \cos\theta d\phi & 0 \end{pmatrix}$$

Using this we can write down the Killing spinor equations on S^3:

$$\partial_\varphi\epsilon - \frac{\sigma^1\sigma^2}{2}\cos\theta\,\epsilon - \frac{\sigma^3\sigma^2}{2}\cos\psi\sin\theta\,\epsilon = \frac{i}{2}\sin\psi\sin\theta\sigma^2\epsilon \tag{A.6}$$

$$\partial_\theta\epsilon - \frac{1}{2}\sigma^3\sigma^1\cos\psi\,\epsilon = \frac{i}{2}\sin\psi\sigma^1\epsilon \tag{A.7}$$

$$\partial_\psi\epsilon = \frac{i}{2}\sigma^3\epsilon \tag{A.8}$$

[1]Note we could have chosen the inequivalent basis $\gamma^{\hat\psi} = -\gamma$. The results should be the same.

Now we make an ansatz for the decomposition of the spinors into linear combinations of $\epsilon^2_\pm$. We write:

$$\epsilon = \sum_a \eta_a \epsilon^2_a \qquad a = \pm$$

and we note that $\epsilon^2_\pm$ satisfy

$$\tilde{\nabla}_\varphi \epsilon^2_a \equiv \partial_\varphi \epsilon^2_a - \frac{\sigma^1\sigma^2}{2}\cos\theta\,\epsilon^2_a = \frac{a}{2}\sigma^3\sigma_{\hat{\varphi}}\sin\theta\,\epsilon^2_a \tag{A.9}$$

$$\tilde{\nabla}_\theta \epsilon^2_a \equiv \partial_\theta \epsilon^2_a = \frac{a}{2}\sigma^3\sigma_{\hat{\theta}}\epsilon^2_a \tag{A.10}$$

where we note that since this is just the derivative on S^2, vielbein-like factor $\sin\psi$ from the warp factor should not be added.

Substituting in this ansatz, and distinguishing between the two subspaces $K_\pm$ in the factor $\pm$, we find that the η_a satisfy

$$-i\frac{a}{2}\sin\theta\,\eta_a + \frac{i}{2}\cos\psi\sin\theta\,\eta_a \pm \frac{1}{2}\sin\psi\sin\theta\,\eta_{-a} = 0 \tag{A.11}$$

$$ia\,\eta_a - i\cos\psi\,\eta_a \mp \sin\psi\,\eta_{-a} = 0 \tag{A.12}$$

although these are the φ and θ components of the Killing spinor equation, we see they are in fact the same equation here.

We can solve this system neatly by arranging the two coefficients into a vector $H \equiv \begin{pmatrix}\eta_+\\ \eta_-\end{pmatrix}$. We think of Pauli matrices $\tau^{1,2,3}$ acting on this vector space. Then we can express the equations as $MH = 0$, where M is given by

$$M_\pm = \tau^3 - \cos\psi\,\mathbb{1} \pm i\sin\psi\,\tau^1\,.$$

The determinant of this matrix vanishes identically, so it is a good projector. Half of its eigenvalues are zero, corresponding to the eigenvector $\begin{pmatrix}\cos\frac{\psi}{2}\\ \pm i\sin\frac{\psi}{2}\end{pmatrix}$. This reproduces the linear combination (A.5.1).

A.6 Bilinears

We want to form bilinears which transform as scalars/vectors/tensors.

First define

$$\bar{\psi} \equiv \psi^\dagger A$$

and

$$\psi^c \equiv B^{-1}\psi^*$$

There are two types of tensor bilinears, formed with either $\bar{\psi}$ or $\bar{\psi^c}$ on the left. For example here are some scalars:

$$\bar{\psi}\chi \qquad \bar{\chi^c}\bar{\psi}$$

These are scalars since both $\bar{\psi}$ and $\bar{\psi^c}$ transform with the inverse Lorentz transformation on the right. To see this, consider an infinitesimal Lorentz transformation, $\Lambda = 1 + \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\delta\bar{\psi} = (\frac{1}{4}\omega^{\mu\nu}\gamma_\mu\gamma_\nu\psi)^\dagger A = \psi^\dagger(-\frac{1}{4}\omega^{\mu\nu}\gamma_\mu^\dagger\gamma_\nu^\dagger)A = \psi^\dagger A(-\frac{1}{4}\omega^{\mu\nu}\gamma_\mu\gamma_\nu)$$

So we have proven the result. To see that the same thing follows for $\bar{\psi^c}$, note that as the Majorana conjugate, ψ^c is designed to transform in the same way as ψ under Lorentz transformations. Taking the bar of this, again it transforms as the inverse.

A.7 Fierz identities

Fierz identities express the fact that the gamma matrices and their antisymmetrized products furnish a complete basis for $\mathcal{S} \times \mathcal{S}$ matrices, where $\mathcal{S}$ is the spinor dimension. Also it is an orthogonal basis with the respect to the trace inner product $A \cdot B \equiv \mathrm{Tr}\,(AB)$. In particular the basis is formed by

$$\mathbb{1} \tag{A.13}$$

$$\gamma^\mu \tag{A.14}$$

$$\gamma^{\mu\nu} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu) \tag{A.15}$$

$$\ldots \tag{A.16}$$

$$\gamma^{\mu_1\mu_2\cdots\mu_n} \equiv \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\,\gamma^{\sigma(\mu_1)}\gamma^{\sigma(\mu_2)}\ldots\gamma^{\sigma(\mu_n)} \tag{A.17}$$

where in even dimensions n goes from 1 to D, and past $D/2$ we can rewrite in terms of the chirality matrix. In odd dimension the matrices past $D/2$ are linear dependent on those before, so n goes from 1 to $[D/2]$. We evaluate the inner products:

$$\mathrm{Tr}\,(\mathbb{1}\mathbb{1}) = \mathcal{S} \tag{A.18}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = \mathcal{S}\eta^{\mu\nu} \tag{A.19}$$

$$\mathrm{Tr}\,(\gamma^{\mu\nu}\gamma^{\alpha\beta}) = -\mathcal{S}\eta^{\mu\nu}\eta^{\alpha\beta} \tag{A.20}$$

$$\ldots \tag{A.21}$$

and all other inner products vanish. The pattern of signs seems to alternate after the first two pluses, between plus and minus.

Lemma: Given a complete orthogonal basis of matrices $\mathcal{O}_I$ satisfying $\mathrm{Tr}\,(\mathcal{O}_I\mathcal{O}_J) = N_I\delta_{IJ}$, we can expand any matrix M in terms of this basis as follows:

$$M = c^I\mathcal{O}_I \quad \text{where} \quad c^I = \frac{1}{N_I}\mathrm{Tr}\,(M\mathcal{O}_I)$$

Using this lemma, we can express any matrix in the Clifford algebra representation as:

$$M = \frac{1}{\mathcal{S}}\left(\mathrm{Tr}\,(M) + \mathrm{Tr}\,(M\gamma_\mu)\gamma^\mu - \frac{1}{2}\mathrm{Tr}\,(M\gamma_{\mu\nu})\gamma^{\mu\nu} + \cdots\right)$$

The $\frac{1}{2}$ factor, which is generalized to $\frac{1}{n!}$ in higher terms, is present because to complete the basis it is sufficient to have only e.g. the $\gamma^{\mu\nu}$ for $\mu < \nu$. More conventional versions of the Fierz identities can all be derived from this.

A.7.1 Example: 4D

A basis of 4×4 matrices is formed by $\mathbb{1}$, γ^μ, $\gamma^{\mu\nu}$, $\gamma_5\gamma^\mu$, and γ_5. The traces are:

$$\mathrm{Tr}\,(\mathbb{1}) = 4 \quad \mathrm{Tr}\,(\gamma^\mu) = 4\eta^{\mu\nu} \quad \mathrm{Tr}\,(\gamma^{\mu\nu}\gamma^{\alpha\beta}) = -4\eta^{\mu\nu}\eta^{\alpha\beta} \quad \mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma_5\gamma^\nu) = -4\eta^{\mu\nu} \quad \mathrm{Tr}\,(\gamma_5\gamma_5) = 4$$

Thus we have, for any 4×4 matrix M:

$$M = \frac{1}{4}\left(\mathrm{Tr}\,(M) + \mathrm{Tr}\,(\gamma_\mu M)\,\gamma^\mu - \frac{1}{2}\mathrm{Tr}\,(\gamma_{\mu\nu}M)\,\gamma^{\mu\nu} - \mathrm{Tr}\,(\gamma_5\gamma_\mu M)\,\gamma_5\gamma^\mu + \mathrm{Tr}\,(\gamma_5 M)\,\gamma_5\right)$$

Using this, we can derive interesting results for the norm of vector and pseudo-vector bilinears, which often arise in supergravity. Define

$$K^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon \quad L^\mu \equiv \bar{\epsilon}\gamma_5\gamma^\mu\epsilon \quad f \equiv \bar{\epsilon}\epsilon \quad \tilde{f} \equiv \bar{\epsilon}\gamma_5\epsilon$$

where we take ϵ to be a commuting spinor. Now $K^2 \equiv K_\mu K^\mu = \bar{\epsilon}\gamma_\mu\epsilon\bar{\epsilon}\gamma^\mu\epsilon$, and we can expand the matrix $\epsilon\bar{\epsilon}$ in our basis. This gives:

$$(\bar{\epsilon}\gamma^\mu\epsilon)^2 = \frac{1}{4}\left(4(\bar{\epsilon}\epsilon)^2 - 2(\bar{\epsilon}\gamma^\mu\epsilon)^2 - 2(\bar{\epsilon}\gamma_5\gamma^\mu\epsilon)^2 - 4(\bar{\epsilon}\gamma_5\epsilon)^2\right)$$

where we have used

$$\gamma_\alpha\gamma^\mu\gamma^\alpha = -2\gamma^\mu \quad \gamma_\alpha\gamma^{\mu\nu}\gamma^\alpha = 0 \quad \gamma_\alpha\gamma_5\gamma^\mu\gamma^\alpha = +2\gamma_5\gamma^\mu \quad \gamma_\alpha\gamma_5\gamma^\alpha = -4\gamma_5$$

so we have:

$$\frac{3}{2}K^2 + \frac{1}{2}L^2 = f^2 - \tilde{f}^2$$

We can do the same expansion for L^2, to obtain

$$\frac{3}{2}L^2 + \frac{1}{2}K^2 = -f^2 + \tilde{f}^2 .$$

We now solve these two simultaneous equations to find that

$$K^2 = -L^2 = f^2 - \tilde{f}^2$$

which is our result. Notice in particular the corollary that if ϵ is chiral ($\gamma_5\epsilon = \pm\epsilon$), then $f = \tilde{f}$, and so $K^2 = L^2 = 0$. This recalls for us the well documented association in 4D between a chiral spinor and a null vector, as utilized for example in the Petrov classification of geometries.

A.8 Spinor projections

In supergravity, the BPS equations for a given ansatz impose projectors on the supersymmetry parameters of the theory.[1] Therefore it is useful to study projection operators acting on spinors. A projector Π is defined as an 'idempotent' linear operator, i.e. one satisfying

$$\Pi^2 = \Pi \tag{A.22}$$

They are typically of the form $\Pi = P \pm 1$, so that the projection condition typically takes the form

$$P\epsilon = \lambda\epsilon \quad \lambda = \pm 1$$

[1]Spinors satisfying these projections, and also satisfying the differential BPS equations, are typically called 'Killing spinors', by analogy with the usual mathematical definition.

where ϵ is a spinor (we take these to be commuting unless explicitly stated otherwise.) For this to have non-trivial solutions, we must have $\det \Pi = \det(P - \lambda)\epsilon = 0$.

A useful consistency condition is $P^2 = \mathbb{1}$. Indeed if we have the equation $P\epsilon = g\epsilon$ for some number g, by acting twice twice with the projector in the appropriate way, we see we must have that g^2, i.e. $g = \lambda = \pm 1$, and we recover the projection above.

A.8.1 Rotated projectors

$$g\lambda\epsilon + x\Gamma^1\epsilon + y\Gamma^2\epsilon = 0 \qquad \lambda = \pm 1 \tag{A.23}$$

where without loss of generality we set $x, y \geq 0$ For this to be a projector, we must have $g^2 = x^2 + y^2$. The killing spinor takes the form $\epsilon = \mathrm{e}^{-\frac{\theta}{2}\Gamma^1\Gamma^2}\epsilon_0$, and we must impose the projection

$$(\Gamma^1 + \lambda)\epsilon_0 = 0$$

Proof: We can write the projection equation as

$$g\lambda\epsilon + \sqrt{x^2 + y^2}\,\mathrm{e}^{\arctan\frac{y}{x}\Gamma^1\Gamma^2}\epsilon = 0$$

We call $\arctan\frac{y}{x} \equiv \theta$. Defining ϵ_0 by $\epsilon = \mathrm{e}^{-\frac{\theta}{2}\Gamma^1\Gamma^2}\epsilon_0$ the equation becomes

$$\mathrm{e}^{-\frac{\theta}{2}\Gamma^1\Gamma^2}(1 + \lambda)\epsilon_0 = 0$$

So that $(1 + \lambda)\epsilon_0 = 0$. Then the equation is satisfied.

A.8.2 Vanishing bilinears

If a spinor ϵ satisfies the projection $H\epsilon = \lambda\epsilon$ (where λ is therefore real), then any bilinear of the form $\epsilon^\dagger M H\epsilon$ vanishes if $\{M, H\} = 0$.

Proof:

$$\epsilon^\dagger M H\epsilon = -\epsilon^\dagger H M\epsilon = -(H\epsilon)^\dagger M\epsilon = -\lambda\epsilon^\dagger M\epsilon = -\epsilon^\dagger M H\epsilon \Rightarrow \epsilon^\dagger M H\epsilon = 0\,.\,\square$$

A.8.3 Projectors involving complex conjugation

These typically arise in the formulation of IIB supergravity in which we have one complex Weyl SUSY parameter ϵ rather than two Majorana-Weyl spinors. Look at the projector

$$M\epsilon^* = \epsilon \tag{A.24}$$

then the consistency condition, akin to $P^2 = \mathbb{1}$ for the normal projectors, is

$$M^* M = \mathbb{1} \tag{A.25}$$

We see this by using (A.24) again inside the LHS of (A.24). Note that we cannot make the projection consistent by e.g. letting $M \mapsto iM$, since this leaves (A.25) invariant.

Example: in order for the Majorana condition $\mathcal{B}^{-1}\epsilon^* = \epsilon$ to be consistent in a given dimension and signature, we must have $(\mathcal{B}^{-1})^*\mathcal{B}^{-1} = \mathbb{1} \Rightarrow \mathcal{B}^*\mathcal{B} = \mathbb{1}$.

Appendix B

Killing spinors of $AdS_5 \times S^5$

Here we quote wholesale from [40] the 32 Killing spinors of the $AdS_5 \times S^5$ solution of IIB supergravity, for reference and in order to make the identification with the superconformal algebra of $\mathcal{N} = 4$. The metric is

$$\mathrm{d}s_{10}^2 = \frac{L^2}{z^2}\left(\mathrm{d}z^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\right) + L^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_4^2\right)$$

where we have chosen a slicing convenient for the brane embeddings in section 1.1. We parametrize S^4 with ϕ_a $a = 1, 2, 3, 4$ in the standard ($\mathrm{d}s_n^2 = \mathrm{d}\phi_{S^n}^2 + \sin^2\phi_n \mathrm{d}s_{S^{n-1}}^2$) coordinates. Then the (32-component) chiral Killing spinors are

$$\epsilon = \left[-z^{1/2}\Gamma_{\hat{z}} h(\theta, \phi_a) + z^{-1/2} h(\theta, \phi_a)\eta_{\mu\nu}x^\mu \Gamma^\nu\right]\eta_2 + z^{-1/2}h(\theta, \phi_a)\eta_1$$

where

$$h(\theta, \phi_a) \equiv \mathrm{e}^{\frac{1}{2}\theta\Gamma_{\hat{4}\hat{5}}}\mathrm{e}^{\frac{1}{2}\phi_1\Gamma_{\hat{5}\hat{6}}}\mathrm{e}^{\frac{1}{2}\phi_2\Gamma_{\hat{6}\hat{7}}}\mathrm{e}^{\frac{1}{2}\phi_3\Gamma_{\hat{7}\hat{8}}}\mathrm{e}^{\frac{1}{2}\phi_4\Gamma_{\hat{8}\hat{9}}}$$

is related to that appearing in [113]. $\eta_{1,2}$ are 10D spinors of opposite chirality

$$\begin{array}{r|l} \Gamma\eta_1 = -\eta_1 & (i\Gamma^{\hat{0}\hat{1}\hat{2}\hat{3}} + 1)\eta_1 = 0 \\ \Gamma\eta_2 = +\eta_2 & (i\Gamma^{\hat{0}\hat{1}\hat{2}\hat{3}} - 1)\eta_2 = 0 \end{array} .$$

Then we parametrize these in terms of two Majorana-Weyl spinors:

$$\eta_1 = \begin{pmatrix} 0 \\ \bar{\epsilon}_s \end{pmatrix} \quad \eta_2 = \begin{pmatrix} \epsilon_c \\ 0 \end{pmatrix}$$

and identify $\epsilon_{s,c}$ as the SUSY generators in section 1.1.

Appendix C

(2+1)D Fierz identities

Here we give, for reference, the Fierz identities relating different bilinears on $\mathcal{M}_3$. In general these can be derived from standard formulae appearing elsewhere, but in our case the analysis is relatively simple, and in practice we derived them with some playing about in Mathematica.

First, we can express the norms of the vectors in terms of the scalars:

$$V_{(0)}^2 = -f^{(1)\,2} - f^{(2)\,2} - f^{(3)\,2} \tag{C.1}$$

$$V_{(1)}^2 = -f^{(0)\,2} + f^{(2)\,2} + f^{(3)\,2} \tag{C.2}$$

$$V_{(2)}^2 = -f^{(0)\,2} + f^{(1)\,2} + f^{(3)\,2} \tag{C.3}$$

$$V_{(3)}^2 = -f^{(0)\,2} + f^{(1)\,2} + f^{(2)\,2} \tag{C.4}$$

and the inner products between them:

$$V^{(I)} \cdot V^{(J)} = -f^{(I)} f^{(J)} \tag{C.5}$$

In three dimensions, we know that not all the $V_{(I)}$, $\tilde{V}_{(I)}$ can be linearly independent - we should be able to express them all in terms of three real vectors, for example $V_{(0,2,3)}$. Indeed there are the relations

$$f^{(0)} V^{(0)} - f^{(1)} V^{(1)} - f^{(2)} V^{(2)} - f^{(3)} V^{(3)} = 0 \tag{C.6}$$

and

$$\frac{f^{(1)}}{\tilde{f}^{(2)}}(f_{(0)}^2 - f_{(1)}^2 - f_{(2)}^2 - f_{(3)}^2)\tilde{V}^{(0)} =$$
$$(if^{(1)}f^{(2)} + f^{(3)}f^{(2)})V^{(0)} - (if^{(2)}f^{(3)} - f^{(3)}f^{(2)})V^{(2)} - (f_{(1)}^2 + f_{(3)}^2)V^{(3)} \tag{C.7}$$

$$\frac{f^{(1)}}{\tilde{f}^{(2)}}(f_{(0)}^2 - f_{(1)}^2 - f_{(2)}^2 - f_{(3)}^2)\tilde{V}^{(1)} =$$
$$(if^{(0)}f^{(2)} + f^{(1)}f^{(3)})V^{(0)} - i(f_{(1)}^2 + f_{(2)}^2)V^{(2)} - (f^{(0)}f^{(1)} - if^{(2)}f^{(3)})V^{(3)} \tag{C.8}$$

$$\frac{f^{(1)}}{\tilde{f}^{(2)}}(f_{(0)}^2 - f_{(1)}^2 - f_{(2)}^2 - f_{(3)}^2)\tilde{V}^{(3)} =$$
$$(f_{(0)}^2 - f_{(1)}^2)V^{(0)} - (if^{(1)}f^{(3)} + f^{(0)}f^{(2)})V^{(2)} + (if^{(1)}f^{(2)} - f^{(0)}f^{(3)})V^{(3)} \tag{C.9}$$

Appendix D

Calibration conditions and equations of motion for the smeared F1-D3 system

We work in Einstein frame in the conventions of [23]. We restrict our discussion to the situation with $F_3 = 0$, and y-dependent density distributions and harmonic functions. Smearing the fundamental strings along all the transverse directions, the Nambu–Goto action plus coupling to the NS form are schematically,

$$S_{\mathrm{F1}} = -\int \left(e^{\frac{\phi}{2}}\,\mathcal{K}_2 - B_2\right) \wedge \Omega_8\,, \tag{D.1}$$

with a particular choice of smearing form

$$\Omega_8 = -\rho_{\mathrm{F1}}(y)\ \mathrm{d}y \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \epsilon_{(4)}\,. \tag{D.2}$$

The function of the radial coordinate $\rho_{\mathrm{F1}}(y)$ describes the string charge distribution. Associated to a string world-sheet embedding, one introduces a calibration form, given essentially by the induced metric on the string,

$$\mathcal{K}_2 = -h_1^{-\frac{3}{4}}\,\mathrm{d}t \wedge \mathrm{d}x\,. \tag{D.3}$$

In the presence of smeared sources the calibration condition, which ensures that the embedding of the branes/strings respects supersymmetry, has to be modified (see for instance [85]). In the case of fundamental strings the modified condition is

$$\mathrm{d}\left(e^{\frac{\phi}{2}}\,\mathcal{K}_2\right) = H_3\,. \tag{D.4}$$

Using that $h_1 = e^{-2\phi}$, one can easily check that this condition is verified in our ansatz, establishing that general backreacting string distributions are consistent with the supersymmetries of the setup. Similarly, we introduce a set of D3 branes extending along the 4D Minkowski directions and smeared on the transverse coordinates , with action

$$S_{\rm D3} \;=\; -\int (\mathcal{K}_4 - 4C_4) \wedge \Omega_6 \,. \tag{D.5}$$

In this case the smearing form is

$$\Omega_6 \;=\; -\rho_{\rm D3}(y)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \epsilon_{(4)}\,, \tag{D.6}$$

while the calibration form reads

$$\mathcal{K}_4 \;=\; -h_3^{-1}\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3\,, \tag{D.7}$$

where again the function $\rho_{\rm D3}(y)$ parametrizes the brane charge along the radial direction. The pertinent calibration condition

$$\mathrm{d}\mathcal{K}_4 \;=\; 4\mathrm{d}C_4 \tag{D.8}$$

is also straightforwardly satisfied, again in line with the expected supersymmetry of the backgrounds.

The presence of these smeared sources alters several equations of motion, that now read

$$\begin{aligned}
&\mathrm{d}\left(e^{-\phi} * H_3\right) + \Omega_8 = 0\\
&\mathrm{d} * F_5 + \tfrac{1}{4}\Omega_6 = 0\\
&\mathrm{d} * \mathrm{d}\phi + \frac{1}{2}e^{-\phi}\,H_3 \wedge *H_3 - \frac{1}{2}e^{\frac{\phi}{2}}\,\mathcal{K}_2 \wedge \Omega_8 = 0\\
&G_{MN} = T^{\rm IIB}_{MN} + T^{\rm F1}_{MN} + T^{\rm D3}_{MN}
\end{aligned} \tag{D.9}$$

The equations for F_1 and F_3 are not modified and turn out to be automatically satisfied within our ansatz. Notice that B_2 couples electrically to the strings through the Nambu–Goto action, so the Bianchi $dH_3 = 0$ remains intact. As is customary the

Bianchi for F_5 coincides with its equation of motion. The stress tensors for the smeared sources are

$$\begin{aligned}
T^{\rm F1}_{MN} &= -\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm F1}}{\delta g^{MN}} = \frac{1}{2}e^{\frac{\phi}{2}}\Big(g_{MN}\,\Omega_8\lrcorner(*\mathcal{K}_2) - \iota_{(M}\Omega_8\lrcorner\iota_{N)}(*\mathcal{K}_2)\Big)\\
T^{\rm D3}_{MN} &= -\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm D3}}{\delta g^{MN}} = \frac{1}{2}\Big(g_{MN}\,\Omega_6\lrcorner(*\mathcal{K}_4) - \iota_{(M}\Omega_6\lrcorner\iota_{N)}(*\mathcal{K}_4)\Big) \qquad \text{(D.10)}
\end{aligned}$$

where for arbitrary p-forms ω_p and ξ_p we have defined the operations

$$\begin{aligned}
\iota_M\omega_p &\equiv \frac{1}{(p-1)!}(\omega_p)_{MN_1\cdots N_{p-1}}\,\mathrm{d}x^{N_1}\wedge\cdots\wedge\mathrm{d}x^{N_{p-1}}\\
\omega_p\lrcorner\xi_p &\equiv \frac{1}{p!}(\omega_p)_{M_1\cdots M_p}(\xi_p)^{M_1\cdots M_p} \qquad \text{(D.11)}
\end{aligned}$$

Using the known ansatz for the forms, all the equations above boil down to the equations determining the harmonic functions sourced by the charge distributions

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y}\left(y^4\frac{\mathrm{d}h_1}{\mathrm{d}y}\right) &= -\rho_{\rm F1}(y)\,,\\
\frac{\mathrm{d}}{\mathrm{d}y}\left(y^4\frac{\mathrm{d}h_3}{\mathrm{d}y}\right) &= -\rho_{\rm D3}(y)\,. \qquad \text{(D.12)}
\end{aligned}$$

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